

EE 313 Linear Signals & Systems

Solution Set for HW#8 on Continuous Time Freq. Resp. & Fourier Trans.

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- Prologue:** This question revisits the Fourier series sum (taught earlier in the course) and asks you to visualize the effect of various filters on a periodic continuous-time signal.

Solution:

a) The Fourier series representation of a periodic signal $x(t)$ with period T is:

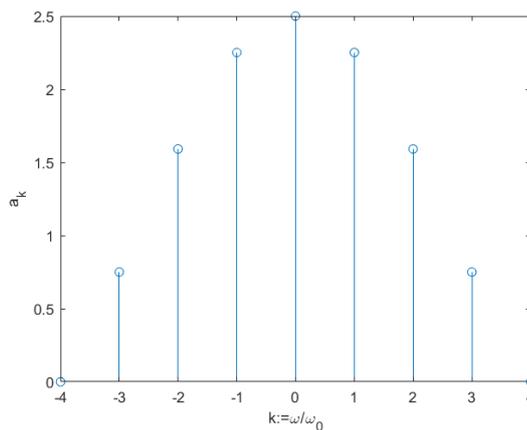
$$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

From the graph, $T = 8$ seconds $\Rightarrow \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$ rad/s.

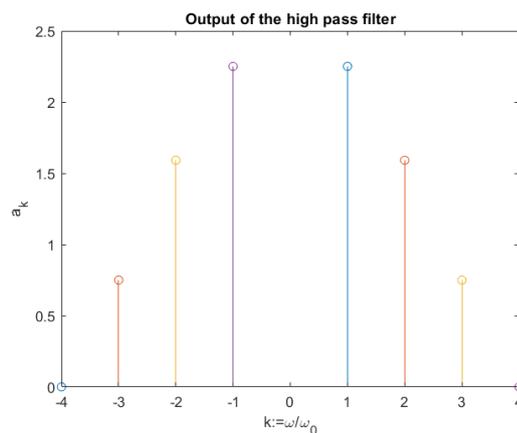
The formulae for evaluating a_k 's are $\begin{cases} a_0 = \frac{1}{8} \int_{-4}^4 x(t) dt = \frac{1}{8} \int_{-1}^1 10 dt \\ a_k = \frac{1}{8} \int_{-4}^4 x(t) e^{-jk\left(\frac{2\pi}{T}\right)t} dt = \frac{1}{8} \int_{-1}^1 10 e^{-jk\left(\frac{2\pi}{T}\right)t} dt \end{cases}$

The a_k terms follow a sinc pattern, i.e. $a_k = 2.5 \sin(\pi k/4) / (\pi k/4)$.

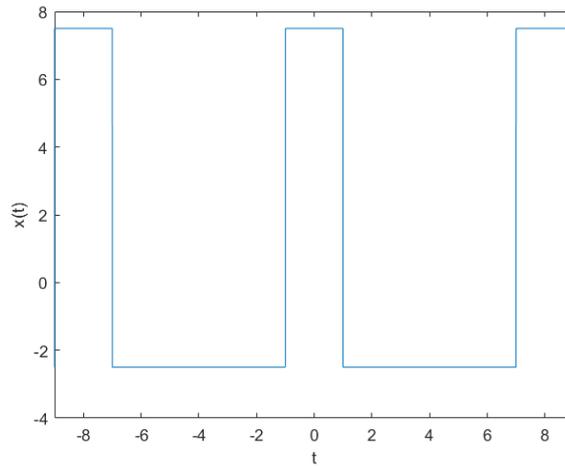
b)



c)



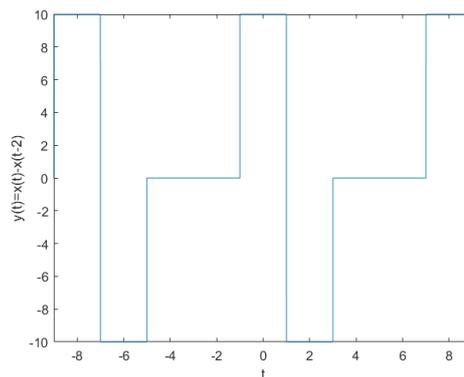
The high pass filter removes the signal component at 0 Hz. This is equivalent to removing the DC component (average value) of the original signal. So, the filtered signal is the original signal shifted down by its average value ($a_0 = 2.5$).



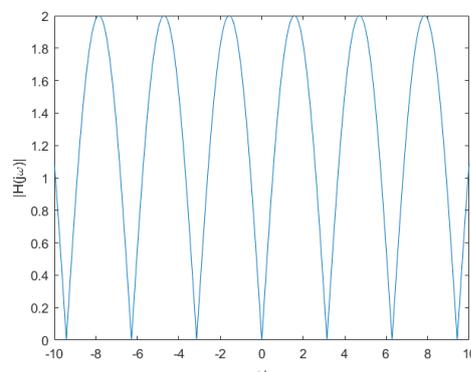
d) For getting an output of the form $y(t) = A + B \cos(\omega_0 t + \phi)$, we'll need a non-zero spectral component at $\omega = 0$, and spectral components at $\omega = \omega_0$ and $\omega = -\omega_0$.

So, to retain the required spectral lines, we want $1 < \frac{\omega_{co}}{\omega_0} \leq 2 \Rightarrow \omega_0 < \omega_{co} < 2\omega_0$.

e) $H(e^{j\omega}) = 1 - e^{-2j\omega} \Rightarrow h(t) = \delta(t) - \delta(t - 2)$ by using the inverse continuous-time Fourier transform. So, $y(t) = x(t) * h(t) = x(t) * (\delta(t) - \delta(t - 2)) = x(t) - x(t - 2)$



f)



The magnitude response has multiple passbands and regularly spaced notches.

Epilogue: As seen in tune-up 11 & the miniproject, the last filter (a.k.a. a comb filter) can be used to simulate the effect of echoes and reverberation on acoustics, electromagnetic and other signals.

2. **Prologue:** This question asks you to use several Fourier transform pairs and properties to evaluate the transforms on several common signals.

Solution:

a) Properties used: $rect\left(\frac{t}{\tau}\right) \leftrightarrow \frac{2}{\omega} \sin\left(\frac{\omega\tau}{2}\right)$

$$u(t+3)u(3-t) = rect\left(\frac{t}{6}\right) \leftrightarrow \frac{2}{\omega} \sin(3\omega)$$

b) Properties used: $\sin(\omega) = (e^{j\omega} - e^{-j\omega})/2j$
 $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$

$$\begin{aligned} \sin(4\pi t) \sin(50\pi t) &= -(e^{j4\pi t} - e^{-j4\pi t})(e^{j50\pi t} - e^{-j50\pi t})/4 \\ \sin(4\pi t) \sin(50\pi t) &\leftrightarrow -\frac{4\pi^2 [(\delta(\omega - 4\pi) - \delta(\omega + 4\pi)) * (\delta(\omega - 50\pi) - \delta(\omega + 50\pi))]}{4} \\ &\leftrightarrow -\pi^2 [\delta(\omega - 54\pi) - \delta(\omega + 46\pi) - \delta(\omega - 46\pi) + \delta(\omega + 54\pi)] \end{aligned}$$

c) Properties used: $\sin(\omega) = (e^{j\omega} - e^{-j\omega})/2j$
 $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$

$$\frac{\sin(\pi t)}{\pi t} \leftrightarrow rect\left(\frac{\omega}{2\pi}\right)$$

$$\begin{aligned} \frac{\sin(4\pi t)}{\pi t} \sin(50\pi t) &= \frac{4 \sin(4\pi t)}{4\pi t} \cdot \frac{e^{j50\pi t} - e^{-j50\pi t}}{2j} \\ &\leftrightarrow 4 \left[rect\left(\frac{\omega}{8\pi}\right) * \frac{\pi}{j} (\delta(\omega - 50\pi) - \delta(\omega + 50\pi)) \right] \\ &\leftrightarrow \frac{4\pi}{j} \left[rect\left(\frac{\omega - 50\pi}{8\pi}\right) - rect\left(\frac{\omega + 50\pi}{8\pi}\right) \right] \end{aligned}$$

d) Properties used: $rect\left(\frac{t}{\tau}\right) \leftrightarrow \frac{2}{\omega} \sin\left(\frac{\omega\tau}{2}\right)$

$$\left(\frac{\sin(200\omega)}{\omega}\right)^2 = \frac{1}{4} \left(\frac{\sin(400(\frac{\omega}{2}))}{\frac{\omega}{2}}\right)^2 \leftrightarrow \frac{1}{4} \left(rect\left(\frac{t}{400}\right) * rect\left(\frac{t}{400}\right) \right)$$

$$\begin{aligned} rect\left(\frac{t}{\tau}\right) * rect\left(\frac{t}{\tau}\right) &= \int_{-\infty}^{\infty} rect\left(\frac{\alpha}{\tau}\right) rect\left(\frac{t-\alpha}{\tau}\right) d\alpha = \int_{-\tau/2}^{\tau/2} rect\left(\frac{t-\alpha}{\tau}\right) d\alpha \\ &= \begin{cases} 0, & \frac{t-\tau/2}{\tau} > \frac{1}{2} \Rightarrow t > \tau \\ \tau - |t|, & -\tau \leq t \leq \tau \\ 0, & \frac{t+\tau/2}{\tau} < -\frac{1}{2} \Rightarrow t < -\tau \end{cases} \end{aligned}$$

Please see Course Handout E on Convolution of Two Rectangular Pulses.

$$\text{So, } \left(\frac{\sin(200\omega)}{\omega}\right)^2 \leftrightarrow \frac{1}{4} \left(rect\left(\frac{t}{400}\right) * rect\left(\frac{t}{400}\right) \right) = \begin{cases} 0, & t > 400 \\ 100 - |t|/4, & -400 \leq t \leq 400 \\ 0, & t < -400 \end{cases}$$

e) Properties used: $\cos(\omega) = (e^{j\omega} + e^{-j\omega})/2$
 $\delta(t - t_0) \leftrightarrow e^{-j\omega t_0}$

$$\begin{aligned} \cos^2(\omega) &= ((e^{j\omega} + e^{-j\omega})/2)^2 \leftrightarrow \pi^2 [(\delta(t+1) + \delta(t-1)) * (\delta(t+1) + \delta(t-1))] \\ &\leftrightarrow \pi^2 [\delta(t-2) + \delta(t+2) + \delta(t) + \delta(t)] = \pi^2 [\delta(t-2) + \delta(t+2) + 2\delta(t)] \end{aligned}$$

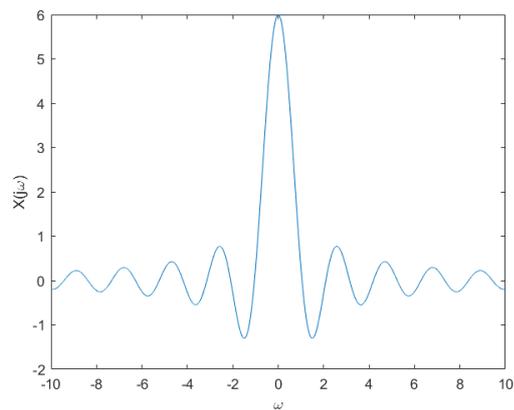
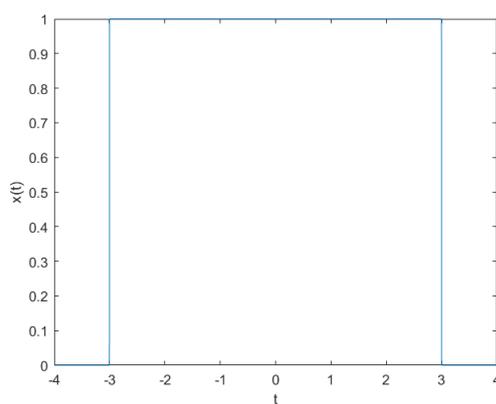
Code:

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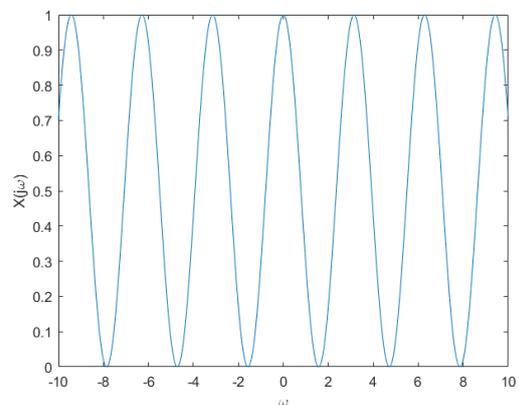
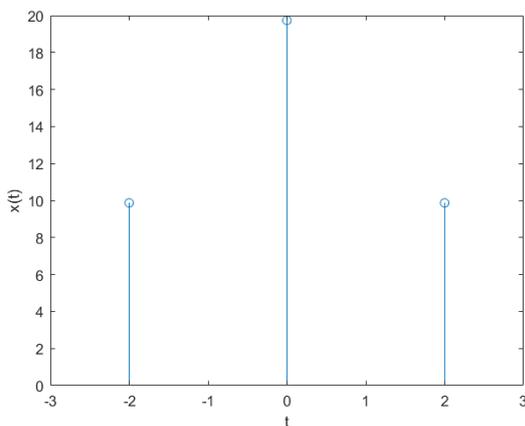
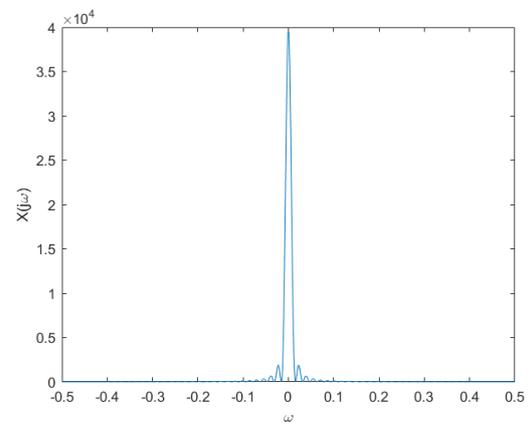
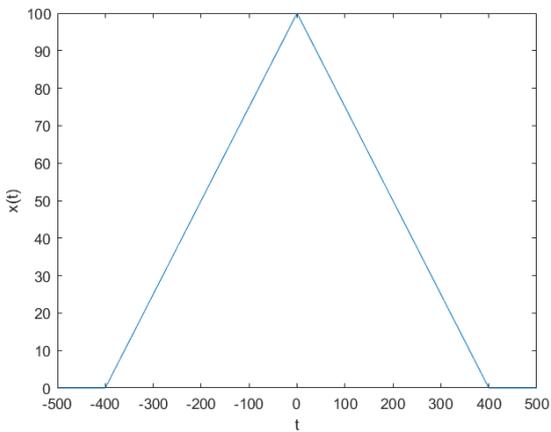
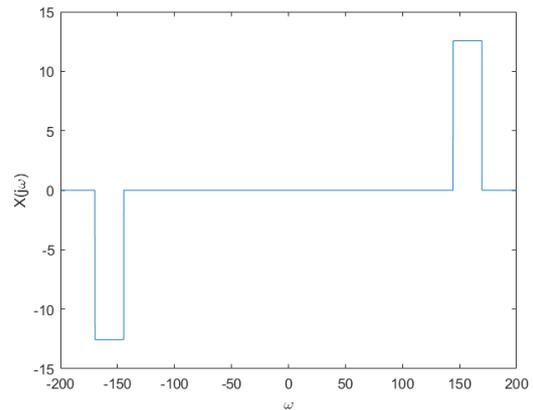
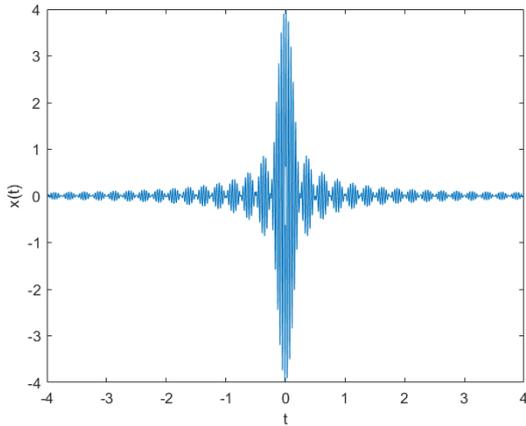
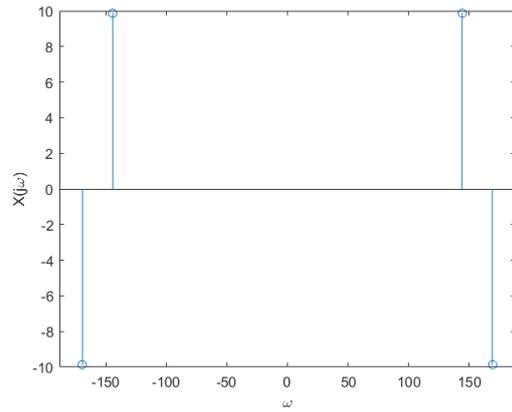
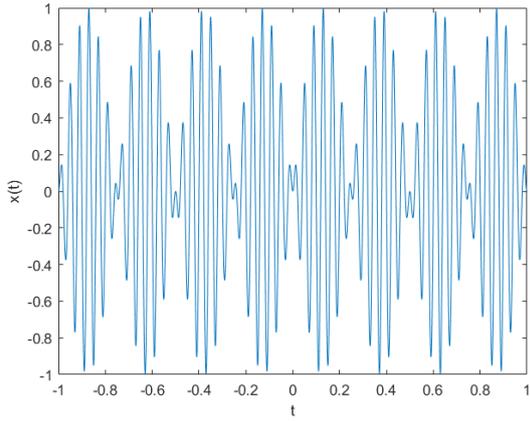
%Q2a)
t=-4:0.001:4;
w = -10:0.001:10;
plot(t,rectpuls(t/6));xlabel('t');ylabel('x(t)');
figure;
plot(w,(2./w).*sin(3.*w));xlabel('\omega');ylabel('X(j\omega)');
figure;
%Q2b)
plot(t,sin(4*pi*t).*sin(50*pi*t));xlabel('t');ylabel('x(t)');xlim([-1,1]);
figure;
w1=[54*pi,-46*pi,46*pi,-54*pi];
m=(-pi^2).*[1,-1,-1,1];
stem(w1,m);xlabel('\omega');ylabel('X(j\omega)');xlim([-60*pi,60*pi]);
figure;
%Q2c)
plot(t,(sin(4*pi*t)./(pi*t)).*sin(50*pi*t));xlabel('t');ylabel('x(t)');
figure;
w1=-200:0.001:200;
plot(w1,4*pi*(rectpuls(w1/(8*pi)-50/8)-
rectpuls(w1/(8*pi)+50/8)));xlabel('\omega');ylabel('X(j\omega)');
figure;
%Q2d)
t1=-500:0.01:500;
x1=100-abs(t1)/4;
x1=x1.*(t1<400 & t1>-400);
plot(t1,x1);xlabel('t');ylabel('x(t)');
figure;
plot(w,(sin(200*w)./w).^2);xlabel('\omega');ylabel('X(j\omega)');xlim([-0.5,0.5]);
figure;
%Q2e)
t1=[-2,0,2];
x1=(pi^2).*[1,2,1];
stem(t1,x1);xlabel('t');ylabel('x(t)');xlim([-3,3]);
figure;
plot(w,cos(w).^2);xlabel('\omega');ylabel('X(j\omega)')

```

For parts (a) through (e), the time-domain plot will be on the left, and the frequency-domain plot will be on the right. Since MATLAB cannot plot Dirac deltas, the frequency domain plot for (b) and the time-domain plot for (e) will be a Fourier series coefficient plots.



Interpret plot on right as a Fourier series that represents Dirac deltas of areas -10, 10, 10 & -10:



Interpret the plot on left as a Fourier series plot having Dirac deltas with areas of π^2 , $2\pi^2$ and π^2 .

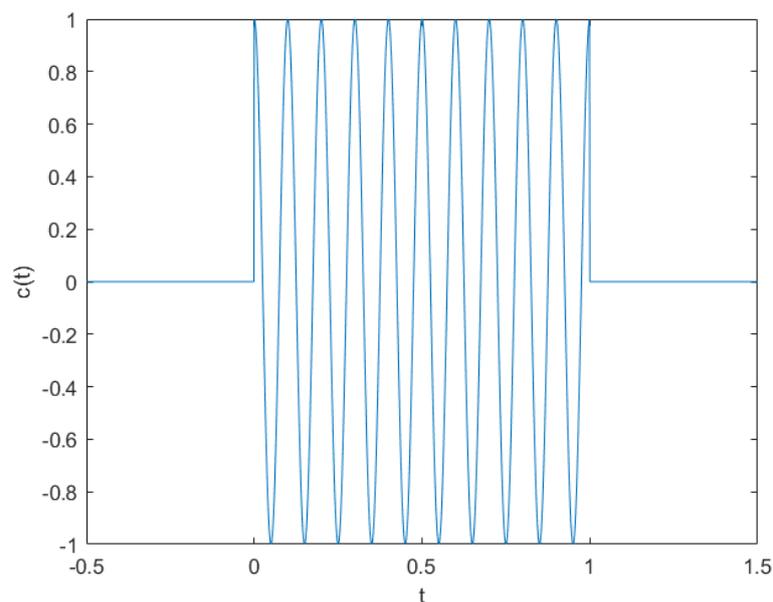
Epilogue: As we can see, a large variety of Fourier transforms can be decomposed down into the “standard” transform pairs. This provides a powerful technique for hand analysis of continuous (and even discrete) time signals.

3. **Prologue:** This question deals with a time limited sinusoid, which is what the idealized sinusoid boils down to in any real-life context.

Solution:

a) Code:

```
t=-0.5:0.001:1.5;
f = 10;
c = cos(2*pi*f*t).*rectpuls(t-0.5);
plot(t,c);
xlabel('t');ylabel('c(t)');
```



$$b) c(t) = \cos(2\pi f_c t) \text{rect}\left(t - \frac{1}{2}\right)$$

Properties used:

$$\cos(2\pi f_c t) = \frac{(e^{j2\pi f_c t} + e^{-j2\pi f_c t})}{2}$$

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

$$\text{rect}(t) \leftrightarrow \frac{2}{\omega} \sin\left(\frac{\omega}{2}\right) = \text{sinc}\left(\frac{\omega}{2}\right) \quad [\text{sinc}(x) := \frac{\sin(x)}{x}]$$

$$x(t) \leftrightarrow X(e^{j\omega}) \Rightarrow x(t - t_0) \leftrightarrow X(e^{j\omega})e^{-j\omega t_0}$$

$$c(t) = \cos(2\pi f_c t) \text{rect}\left(t - \frac{1}{2}\right) = \frac{1}{2} \cdot ((e^{j2\pi f_c t} + e^{-j2\pi f_c t}) \text{rect}\left(t - \frac{1}{2}\right))$$

$$\leftrightarrow \frac{1}{2} \cdot 2\pi(\delta(\omega - 2\pi f_c) + \delta(\omega + 2\pi f_c)) * \left(\text{sinc}\left(\frac{\omega}{2}\right) e^{-\frac{j\omega}{2}}\right)$$

$$\leftrightarrow \pi(\delta(\omega - 2\pi f_c) + \delta(\omega + 2\pi f_c)) * \left(\text{sinc}\left(\frac{\omega}{2}\right) e^{-\frac{j\omega}{2}}\right)$$

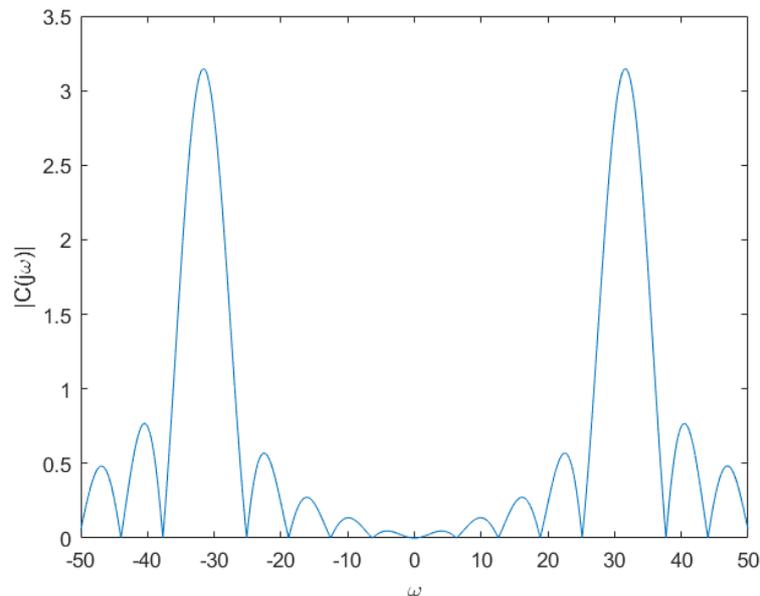
$$\leftrightarrow \pi \left[\left(\text{sinc}\left(\frac{\omega - 2\pi f_c}{2}\right) e^{-\frac{j2\pi f_c}{2}} + \text{sinc}\left(\frac{\omega + 2\pi f_c}{2}\right) e^{\frac{j2\pi f_c}{2}} \right) \right]$$

$$\leftrightarrow \pi \left[\left(\text{sinc} \left(\frac{\omega}{2} - \pi f_c \right) e^{-j\pi f_c} + \text{sinc} \left(\frac{\omega}{2} + \pi f_c \right) e^{j\pi f_c} \right) \right]$$

$$\text{So, } \cos(2\pi f_c t) \text{rect} \left(t - \frac{1}{2} \right) \leftrightarrow \pi \left[\left(\text{sinc} \left(\frac{\omega}{2} - \pi f_c \right) e^{-j\pi f_c} + \text{sinc} \left(\frac{\omega}{2} + \pi f_c \right) e^{j\pi f_c} \right) \right].$$

c) Code:

```
fc = 5;
w = -50:0.001:50;
Cjw = pi.*(sinc(w/(2*pi)-fc).*exp(-j*pi*fc)+sinc(w/(2*pi)+fc).*exp(j*pi*fc));
plot(w,abs(Cjw));
xlabel('\omega');ylabel('|C(j\omega)|');
```



d) 1st null to right of $\omega_c = 37.73$ rad/s

1st null to left of $\omega_c = 25.09$ rad/s

So, bandwidth = 12.64 rad/s .

Fourier transform of rectangular pulse is a sinc pulse with a null bandwidth of 2π rad/s.

Fourier transform of the modulated rectangular pulse is a sinc pulse centered at ω_c and another sinc pulse centered at $-\omega_c$. The null-to-null bandwidth is 4π .

e) The magnitude of the Fourier transform of a two sided cosine consists of 2 Dirac deltas – one at ω_c , and the other at $-\omega_c$.

Since each of the Dirac deltas has an extent of zero, the bandwidth is 0, i.e. the entire power of the signal is contained in a single frequency.

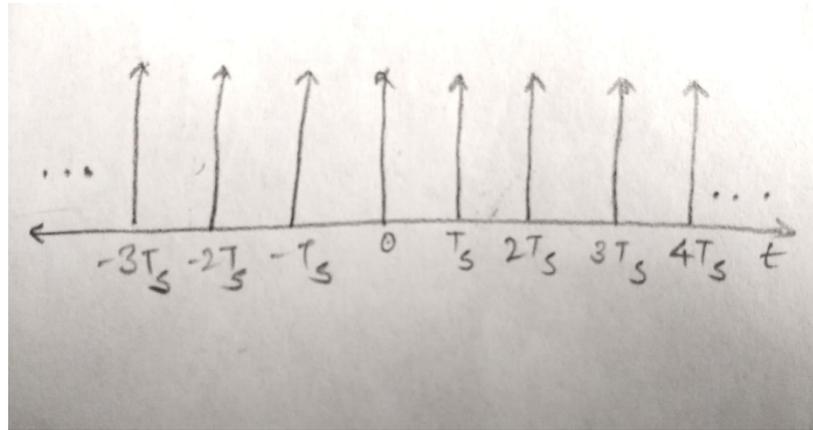
From part (d), we see that the Fourier transform of $c(t)$ has a non zero bandwidth. This is because the multiplication with $\text{rect}(t-0.5)$ limits the signal in the time domain. This results in a “smearing” of the signal power across a band of frequencies centred around f_c .

Epilogue: Here, we see that the time limited sinusoid is a pretty decent approximation to the infinite duration sinusoid in the frequency domain. Increasing the sinusoid duration (i.e. using something like $\cos(2\pi f_c t) \text{rect} \left(\frac{t}{k} \right)$, $k > 1$) reduces the width of the main lobes, hence improving the approximation accuracy in the frequency domain.

4. **Prologue:** This question introduces the impulse train, a.k.a. the Dirac comb. The signal introduced here is used for ideal sampling.

Solution:

- a) The time-domain plot of an impulse train $p(t) = \sum \delta(t - nT_s)$ with spacing between impulses of T_s and the area under each impulse (Dirac delta) is 1:



b) $p(t) = \sum \delta(t - nT_s)$

Clearly, this is a pulse train with an impulse at every multiple of T_s . So, the period of this signal is T_s .

c) $p(t) = \frac{1}{T_s} (1 + 2 \cos(\omega_s t) + 2 \cos(2\omega_s t) + \dots)$

Approach #1: Use Fourier transform pairs $1 \leftrightarrow 2\pi \delta(\omega)$ and $\cos(\omega_s t) \leftrightarrow \pi \delta(\omega + \omega_s) + \pi \delta(\omega - \omega_s)$ to obtain $P(j\omega) = \frac{1}{T_s} (2\pi + 2\pi\delta(\omega + \omega_s) + 2\pi\delta(\omega - \omega_s) + 2\pi\delta(\omega + 2\omega_s) + 2\pi\delta(\omega - 2\omega_s) + \dots) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$

Approach #2:

$$p(t) = \frac{1}{T_s} (1 + e^{j\omega_s t} + e^{-j\omega_s t} + e^{2j\omega_s t} + e^{-2j\omega_s t} + \dots)$$

$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

Using the Fourier transform pair $e^{j\omega_0 t} \leftrightarrow 2\pi \delta(\omega - \omega_0)$, we get:

$$P(e^{j\omega}) = \frac{1}{T_s} (2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)) = \omega_s (\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)).$$

- d) As we can see from the formula above, the spacing between the impulses is $\omega_s \frac{\text{rad}}{s}$.

Epilogue: The Fourier transform of an impulse train in the time domain is an impulse train in the Fourier domain. This transform pair forms the mathematical basis of sampling, and can be used to derive the sampling & reconstruction theorems (Nyquist sampling theorem & sinc interpolation) taught throughout the semester. Please see *Signal Processing First Sec. 12-3*.