EE 313 Linear Signals & Systems

Solution Set for HW#9 on Laplace Transforms

By: Anyesha Ghosh & Prof. Brian L. Evans

1. Given the Laplace transform property $L\big(x(t-t_d)\big) = e^{-st_d}L(x(t))$ where the region of convergence is the region of convergence of $L(x(t))$.

   a) $x(t) = \delta(t+1) + 2\delta(t) + \delta(t-1)$

   $L\big(\delta(t)\big) = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1$ and the region of convergence is the entire *s* plane.

   So, $L\big(x(t)\big) = L\big(\delta(t+1)\big) + 2L\big(\delta(t)\big) + L\big(\delta(t-1)\big) = e^{s} + 2 + e^{-s}$

   Region of convergence is the entire s plane.

   To check the answer, one could apply the Laplace transform definition to *x*(*t*).

   b) $x(t) = 2e^{-t}u(t-5)$

   $x(t) = 2e^{-(t-5+5)}u(t-5) = 2e^{-5}e^{-(t-5)}u(t-5)$
   From lecture, $L\big(e^{-at}u(t)\big) = \frac{1}{s+a}$ and the region of convergence is Re{*s*} > -Re{*a*}.
   So, $L\big(x(t)\big) = \frac{2e^{-5}e^{-5s}}{s+1}$ and the region of convergence is Re{*s*} > -Re{*a*}.

   c) $x(t) = e^{-t}[u(t) - u(t-4)]$
   $x(t) = e^{-t}u(t) - e^{-(t-4+4)}u(t-4) = e^{-t}u(t) - e^{-4}e^{-(t-4)}u(t-4)$
   From lecture, $L\big(e^{-at}u(t)\big) = \frac{1}{s+a}$ and the region of convergence is Re{*s*} > -Re{*a*}.
   So, $L\big(x(t)\big) = \frac{1}{s+1} - \frac{e^{-4}e^{-4s}}{s+1} = \frac{\left(1-e^{-4(1+s)}\right)}{s+1}$

2.

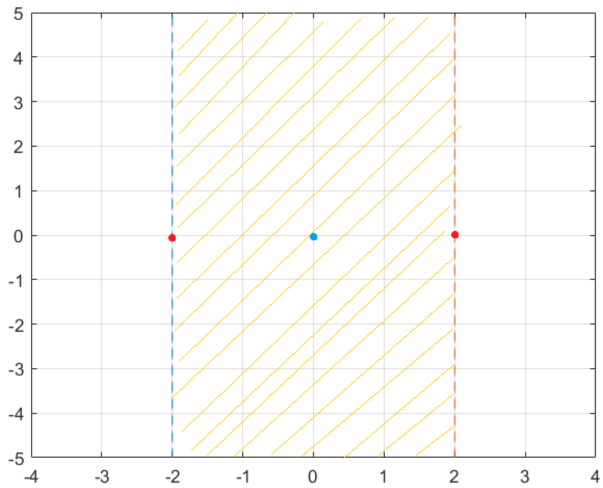
$$x(t) = e^{-2|t|} = \begin{cases} e^{-2t}, t \geq 0 \\ e^{2t}, \ \ t < 0 \end{cases}$$

So, the bilateral s-transform of $x(t) := X(s)$ is,

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{0} e^{2t}e^{-st}\,dt + \int_{0}^{\infty} e^{-2t}e^{-st}\,dt$$

$$= \int_{-\infty}^{0} e^{(2-s)t}\,dt + \int_{-\infty}^{0} e^{-(2+s)t}\,dt = \frac{1}{2-s} + \frac{1}{2+s} = \frac{4}{4-s^2}.$$

So, $X(s) = \frac{4}{4-s^2}$ and the region of convergence (ROC) is determined below.
The ROC of this equation is determined as:
$ROC_1\ (for\ the\ left\ sided\ integral) \coloneqq\ 2 - Re\{s\} > 0 \Rightarrow Re\{s\} < 2.$
$ROC_2\ (for\ the\ right\ sided\ integral) \coloneqq\ 2 + Re\{s\} > 0 \Rightarrow Re\{s\} > -2.$
$So, ROC = ROC_1 \cap ROC_2 = \ -2 < Re\{s\} < 2.$



Pole-zero plot for the Laplace transform. ROC is shaded in yellow. The red dots denote the poles at s = 2 and s = -2. There are no zeroes. The blue dot denotes the origin.

3.

A continuous-time system with input $x(t)$ and output $y(t)$ is governed by the differential equation $\frac{d}{dt}y(t) + 2y(t) = x(t)$. Initial condition $y(0^-) = 0$ so that the system is LTI.

Using the property: $L\left(\frac{d}{dt}x(t)\right) = sL\big(x(t)\big)$, we get:

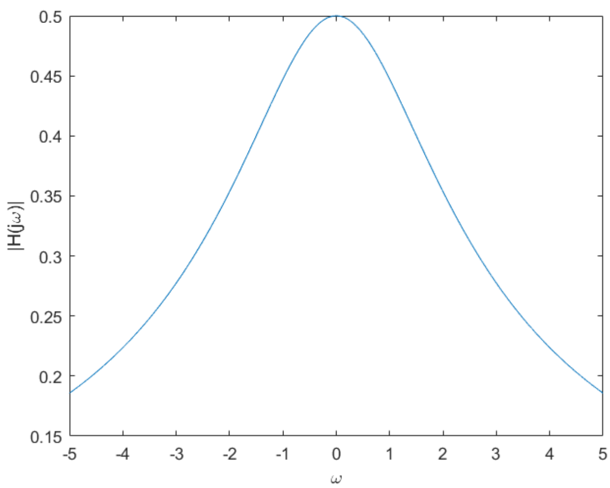
a) $sY(s) + 2Y(s) = X(s) \Rightarrow Y(s)(s+2) = X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+2}$

Because the system is causal, the region of convergence is Re{s} > -2.

b) Using the Laplace transform pair $L\big(e^{-at}u(t)\big) = \frac{1}{s+a}$ for Re{s} > -2, we obtain $h(t) = e^{-2t}u(t)$

c) $H(j\omega) = \frac{1}{j\omega+2}$ by substituting $s = j\omega$ into $H(s)$ above. This substitution is valid because the imaginary axis lies within the region of convergence of Re{s} > -2.

d)



This is a lowpass filter.

e) $x(t) = u(t) => X(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_0^{\infty} e^{-st}dt = \frac{1}{s}$ for Re{s} > 0. We could have also obtained the transform by using $L(e^{-at}u(t)) = \frac{1}{s+a}$ and substituting *a* = 0.

$$Y(s) = H(s)X(s) = \frac{1}{s(s+2)}$$

f) $Y(s) = \frac{1}{s(s+2)} = \frac{1}{2}[\frac{1}{s} - \frac{1}{s+2}]$

Using the Laplace transform pair $L(e^{-at}u(t)) = \frac{1}{s+a}$, for we get

$y(t) = 0.5[u(t) - e^{-2t}u(t)]$.

4.

a) $H(s) = \frac{s-2}{s+2} = \frac{s}{s+2} - \frac{2}{s+2}$.

Using the Laplace transform pairs $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}u(t), L^{-1}(sX(s)) = \frac{d}{dt}x(t)$, we get

$$h(t) = \frac{d}{dt}(e^{-2t}u(t)) - 2e^{-2t}u(t) = -2e^{-2t}u(t) + e^{-2t}\delta(t) - 2e^{-2t}u(t)$$

$$h(t) = e^{-2t}\delta(t) - 4e^{-2t}u(t)$$

Alternate approach: One could put the transfer function into proper fractional form by dividing the denominator into the numerator to get $H(s) = \frac{s-2}{s+2} = 1 - \frac{4}{s+2}$ which has an inverse transform of $h(t) = \delta(t) - 4e^{-2t}u(t)$

b) $H(s) = \frac{s-2}{s+2} => \frac{Y(s)}{X(s)} = \frac{s-2}{s+2}$

$sY(s) + 2Y(s) = sX(s) - 2X(s)$

Using the Laplace transform pair $L^{-1}(sX(s)) = \frac{d}{dt}x(t)$, we get,

$$\frac{d}{dt}y(t) + 2y(t) = \frac{d}{dt}x(t) - 2x(t).$$

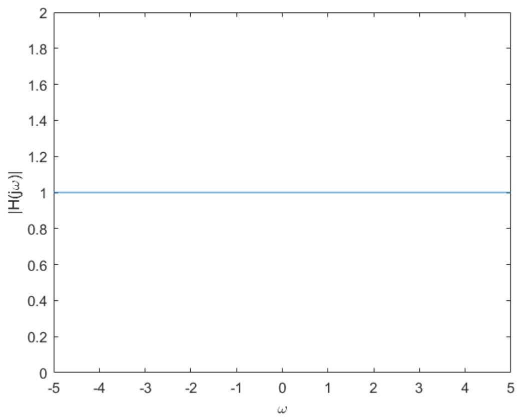
c) $h(t) = -4e^{-2t}u(t) + e^{-2t}\delta(t)$. For bounded-input bounded-output (BIBO) stability,

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} |-4e^{-2t}u(t) + e^{-2t}\delta(t)|dt < \int_{-\infty}^{\infty} 4e^{-2t}u(t)dt + \int_{-\infty}^{\infty} e^{-2t}\delta(t)dt < \infty$$

So, the system is BIBO stable.

$$H(j\omega) = \frac{j\omega - 2}{j\omega + 2} => |H(j\omega)| = \left|\frac{j\omega - 2}{j\omega + 2}\right| = \frac{\sqrt{(-2)^2 + \omega^2}}{\sqrt{2^2 + \omega^2}} = \frac{\sqrt{2^2 + \omega^2}}{\sqrt{2^2 + \omega^2}} = 1.$$

d)



This is an all-pass filter.