

Solution Set for Homework #1 on Sinusoidal Signals

By Mr. Houshang Salimian and Prof. Brian L. Evans

September 7, 2018

1. **Prologue:** This problem helps you to identify the points of interest in a sinusoidal signal, and calculate the parameters of the waveform based on your observations. It relies on the definitions given in Sec. 2-3 of the *Signal Processing First* textbook.

Solution:

To estimate the frequency, we would first estimate the period and then invert the period. We could estimate a period in the plot by measuring the amount of time it takes to go from a peak to the next peak, or from one valley to the next. The plot has valleys at times $t_0 = -1.5$ ms, $t_1 = 0.75$ ms, $t_2 = 3$ ms. One period has elapsed from t_0 to t_1 and two periods have elapsed from t_0 to t_2 . The period is about 2.25 ms long.

$$\omega_0 = 2\pi/T = 888.88\pi = 2792.53 \text{ rad/s.}$$

To calculate A , consider the peak-to-peak amplitude (X_{pp}) of the waveform which is the difference between the maximum value and the minimum value:

$$X_{pp} = 4 - (-4) = 8$$

$$\text{So, } A = X_{pp}/2 = 4.$$

We are left with one unknown, ϕ . We can pick any point in time to give us one equation in one unknown: $x(t) = A \cos(\omega_0 t + \phi)$

- **Approach #1:** If we pick $t = 0$, then $x(0) = 4 \cos(\phi) \approx 2$ and

$$\phi \approx \cos^{-1}(2/4) = 1.0472 \text{ rad} \quad (1)$$

Recall that $\cos(-\phi) = \cos(\phi)$, so ϕ could be $+1.0472$ rad or -1.0472 rad. Also, any addition of ϕ and a multiple of 2π would also be a valid answer.

To decide between the values, consider the slope at $t = 0$, which is negative. (2)

$$x(t) = A \cos(\omega_0 t + \phi) \Rightarrow x'(t) = -A \omega_0 \sin(\omega_0 t + \phi) \Rightarrow x'(0) = -A \omega_0 \sin(\phi)$$

So, condition (2) implies that $x'(0) < 0 \Rightarrow \sin(\phi) > 0$.

Out of the two values in (1), only $\phi = 1.0472$ rad satisfies this condition.

- **Approach #2.** Pick $t = 0.75$ ms where there is a valley. Then,

$$x(0.75\text{ms}) = 4 \cos(2\pi(0.75\text{ms}/2.25\text{ms}) + \phi) = -4 \Rightarrow \cos(2\pi/3 + \phi) = -1 \Rightarrow$$

$$2\pi/3 + \phi = \pi \Rightarrow \phi = \pi/3 = 1.0472 \text{ rad}$$

Epilogue: There is nothing special about the time points considered for the solution. The points were chosen due to convenience, and the problem could have been done by taking other points as well. Additionally, the value of the phase ϕ is not unique. If you find your answers for ω_0 and ϕ different from above calculations, it might be because of observation and approximation errors. The original formula that is used to plot the figure is:

$$x(t) = 4 \cos(2\pi(440) t + \pi/3)$$

2. **Prologue:** This problem helps you to calculate the summation of several sinusoids having the same frequency, but with different amplitudes and phases. Using phasor addition rule the answer can be derived:

part a)

$$x(t) = 2 \sin(\omega_0 t + 45^\circ) + \cos(\omega_0 t) = 2 \sin(\omega_0 t + \pi / 4) + \cos(\omega_0 t)$$

$$\sin(\theta) = \cos(\theta - \pi / 2) \quad (*)$$

by applying (*) in x(t):

$$x(t) = 2 \cos(\omega_0 t + \pi / 4 - \pi / 2) + \cos(\omega_0 t) = 2 \cos(\omega_0 t - \pi / 4) + \cos(\omega_0 t)$$

Using the phasor addition rule from section 2-6.2 of the textbook:

$$\text{Re}\{2e^{j(\omega_0 t - \pi/4)} + e^{j\omega_0 t}\} = \text{Re}\{2e^{-j\pi/4} e^{j\omega_0 t} + e^{j\omega_0 t}\} = \text{Re}\{(2e^{-j\pi/4} + 1)e^{j\omega_0 t}\} = \text{Re}\{Ae^{j\phi} e^{j\omega_0 t}\}$$

Using Euler's formula:

$$2e^{-j\pi/4} = 2 \cos(\pi / 4) - j2 \sin(\pi / 4) = \sqrt{2} - j\sqrt{2}$$

Therefore:

$$Ae^{j\phi} = 2e^{-j\pi/4} + 1 = (\sqrt{2} + 1) - j\sqrt{2} = 2.7979e^{-j0.5299}$$

$$A = 2.7979$$

$$\phi = -0.5299 \text{ (rad)}$$

$$x(t) = \text{Re}\{2.7979e^{j(\omega_0 t - 0.5299)}\} = 2.7979 \cos(\omega_0 t - 0.5299)$$

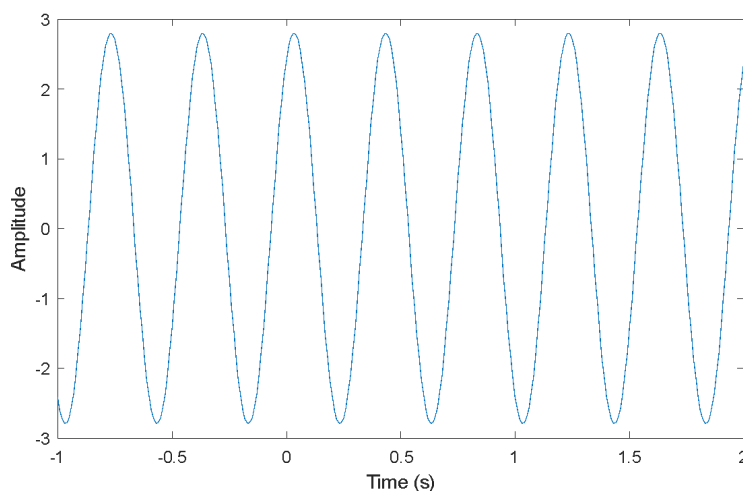
part b) T_0 is the fundamental period of $x(t)$

$$\omega_0 = 2\pi/T_0 = 5\pi$$

$$T_0 = (2\pi)/(5\pi) = 0.4 \text{ s}$$

This range, e.g. $-1 \leq t \leq 2$, covers $T_1 = 2 - (-1) = 3$ (s) of the time scale

$m = T_1 / T_0 = 3 / 0.4 = 7.5$, where $m = 7.5$ is the number of periods that are included in the plot.



Following is the MATLAB code for part (b). We will compare the original equation for $x(t)$ vs. part (a).

```
w0 = 5*pi;
f0 = w0/(2*pi); %fundamental period in Hz
fs = 40*f0; %Sampling frequency in Hz
Ts = 1/fs; %Sampling time in seconds
t = -1:Ts:2;
x = 2*sin(w0*t+pi/4)+cos(w0*t);
plot(t,x);
xlabel('Time (s)');
ylabel('Amplitude');
figure;
xParta = 2.7979 * cos(w0*t - 0.5299);
plot(t, xParta);
```

The two plots are identical.

part c) The answer is calculated in part a.

$$z(t) = 2.7979 \cos(\omega_0 t - 0.5299)$$

3. **Prologue:** This question shows application of Euler's formula to derive imaginary part of a complex signal.

part a) Using Euler's formula:

$$s(t) = 5e^{j\pi/3} e^{j10\pi t} = 5 \cos(10\pi t + \pi/3) + j5 \sin(10\pi t + \pi/3)$$

$$s_i(t) = \text{Im}\{s(t)\} = 5 \sin(10\pi t + \pi/3)$$

$$\omega_0 = 2\pi / T_0 = 10\pi$$

$$T_0 = 2\pi / (10\pi) = 0.2$$

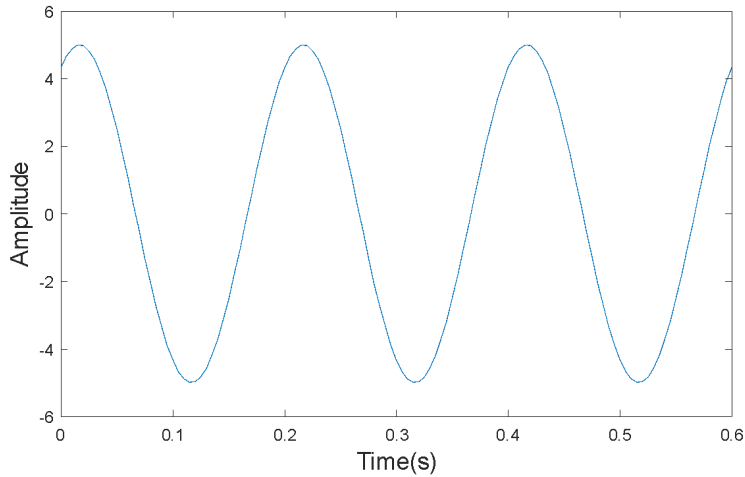
Therefore, to include three periods of fundamental period the time range is :

$$T = 3 * T_0 = 0.6 \text{ s}$$

MATLAB code:

```
w0 = 10*pi;
T0 = 2*pi/(10*pi); %fundamental period
fs = 40/T0; %sampling frequency
Ts = 1/fs; %sampling period
t = 0:Ts: 3*T0; %time range equal to three fundamental periods
s = 5*exp(j*pi/3)*exp(j*w0*t);
si = imag(s); %imaginary part of s(t)
plot(t,si)
xlabel('Time (s)')
ylabel('Amplitude')
```

Using the MATLAB code in the problem, here's the resulting plot of the imaginary part of s(t):



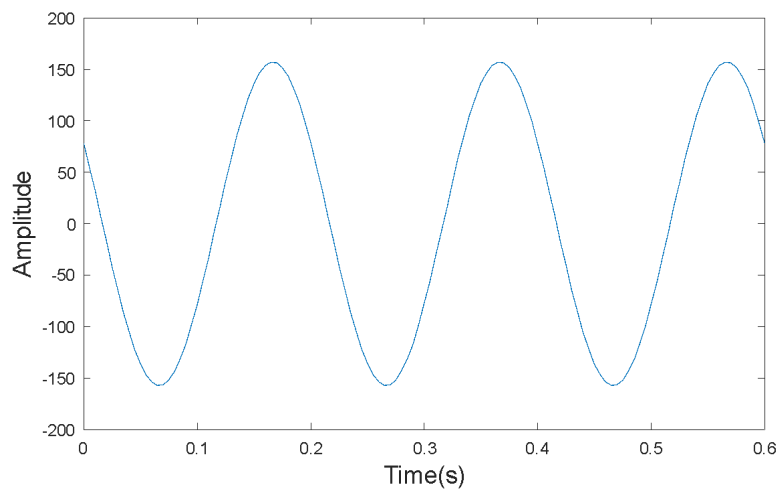
part b)

$$s(t) = j10\pi(5e^{j\pi/3}e^{j10\pi t}) = j50\pi e^{j\pi/3}e^{j10\pi t} = j50\pi\{\cos(10\pi t + \pi/3) + j\sin(10\pi t + \pi/3)\} = -50\pi \sin(10\pi t + \pi/3) + j50\pi \cos(10\pi t + \pi/3)$$

$$q(t) = 50\pi \cos(10\pi t + \pi/3)$$

MATLAB code:

```
w0 = 10*pi;
T0 = 2*pi/(10*pi); %fundamental period
fs = 40/T0; %sampling frequency
Ts = 1/fs; %sampling period
t = 0:Ts: 3*T0; %time range equal to three periods of
fundamental period
s = 5*exp(j*pi/3)*exp(j*w0*t);
sdot = j*w0*s; %differentiation with respect to time
q = imag(sdot); %imaginary part of sdot(t)
plot(t,q)
xlabel('Time (s)')
ylabel('Amplitude')
```



Epilogue: This code shows a particular method of generating a sinusoid in MATLAB. A sinusoid can also be generated by plotting the cosine (or sine) for a given array of time points.

4. **Prologue:** In this problem, a discrete-time signal $x[n]$ is input into a system, and the system produces output signal $y[n] = x[n+1] - 2x[n] + x[n-1]$ for all n . One can directly substitute the formula for $x[n]$ into the formula $y[n] = x[n+1] - 2x[n] + x[n-1]$ for all n and simplify.

Solution:

$$x[n] = 7e^{j(0.22\pi n - 0.25\pi)}$$

$$y[n] = 7e^{j(0.22\pi(n+1) - 0.25\pi)} - 2 \times 7e^{j(0.22\pi n - 0.25\pi)} + 7e^{j(0.22\pi(n-1) - 0.25\pi)} =$$

$$7e^{j(0.22\pi n + 0.22\pi - 0.25\pi)} - 14e^{j(0.22\pi n - 0.25\pi)} + 7e^{j(0.22\pi n - 0.22\pi - 0.25\pi)} =$$

$$7e^{-j0.25\pi} (e^{j0.22\pi} - 2 + e^{-j0.22\pi}) e^{j0.22\pi n} = 3.2128e^{j2.3562} e^{j0.22\pi n}$$

$$y[n] = 3.2128e^{j(0.22\pi n + 2.3562)}$$

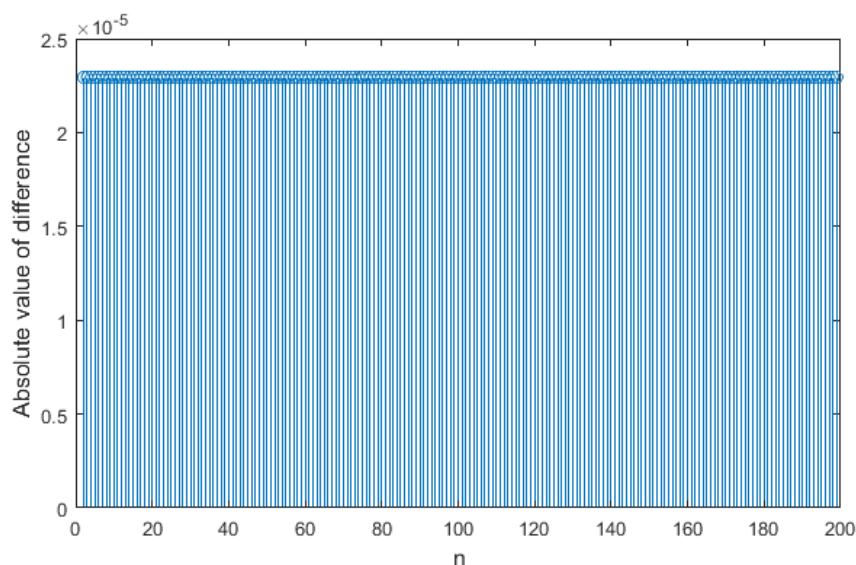
$$A = 3.2128$$

$$\omega_0 = 0.22\pi$$

$$\phi = 2.3562$$

MATLAB code:

```
n = 1:200;
x = 7*exp(j*(0.22*pi*n-0.25*pi));
n = 2:199;
y1 = x(n+1) - 2*x(n) + x(n-1);
y2 = 3.2128*exp(j*(0.22*pi*n+2.3562));
absdiff = abs(y1 - y2);
stem(n, absdiff);
xlabel('n');
ylabel('Absolute value of difference');
```



5. **Prologue:** This problem changes the signal representation from the time domain to the frequency domain. It also deals with periodicity while adding sinusoidal signals, which is explained in Sec. 3-3 of the *Signal Processing First* textbook.

Solution:

$$a) \quad x(t) = 10 + \frac{20e^{j\pi/4} e^{j2\pi(100)t} - 20e^{-j\pi/4} e^{-j2\pi(100)t}}{2} + \frac{10e^{j2\pi(250)t} - 10e^{-j2\pi(250)t}}{2}$$

From part (b), $f_1 = \text{gcd}(100, 250) = 50$ Hz

$\{(f_0 = 0, A_0 = 10), (f_1 = 50, a_1 = 0), (f_{-1} = -50, a_{-1} = 0), (f_2 = 100, a_2 = 10e^{j\pi/4}), (f_{-2} = -100, a_{-2} = 10e^{-j\pi/4}), (f_3 = 150, a_3 = 0), (f_{-3} = -150, a_{-3} = 0), (f_4 = 200, a_4 = 0), (f_{-4} = -200, a_{-4} = 0), (f_5 = 250, a_5 = 5), (f_{-5} = -250, a_{-5} = 5)\}$

- b) We can find all of the values of the period T in seconds, and then identify the period with the smallest positive value as the *fundamental period*. A signal $x(t)$ is periodic with period T if $x(t + T) = x(t)$ for all values of t :

$$x(t) = 10 + 20 \cos(2\pi(100)t + \pi/4) + 10 \cos(2\pi(250)t)$$

$$\begin{aligned} x(t+T) &= 10 + 20 \cos(2\pi(100)(t+T) + \pi/4) + 10 \cos(2\pi(250)(t+T)) \\ &= 10 + 20 \cos(2\pi(100)t + 2\pi(100)T + \pi/4) + 10 \cos(2\pi(250)t + 2\pi(250)T) \end{aligned}$$

In order for $x(t + T) = x(t)$ for all values of t , we can equate the first terms in $x(t + T)$ and $x(t)$, equate the second terms in $x(t + T)$ and $x(t)$, and equate the third terms in $x(t + T)$ and $x(t)$, and find values of T that work for all three terms.

The first terms are already equal to each other (10). The second terms are equal when $2\pi(100)T = 2\pi m$ where m is an integer; i.e., when T is $1/100, 2/100, 3/100, 4/100, 5/100$, etc. The third terms are equal when $2\pi(250)T = 2\pi k$ where k is an integer; i.e., when T is $1/250, 2/250, 3/250, 4/250, 5/250, \dots$. The common values for the period T in seconds among all three terms are $1/50, 2/50, 3/50$, etc., which means that the fundamental period is 50 s, and the fundamental frequency is 50 Hz.

Note that the fundamental frequency is $\text{gcd}(100 \text{ Hz}, 250 \text{ Hz}) = 50 \text{ Hz}$, which comes from the the analysis in the previous paragraph.

- c) Spectrum of signal:

