EE 313 Linear Signals & Systems (Fall 2018)

Solution Set for Homework #10 on Laplace Transforms

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Problem 1.

a) $x(t) = u(t) - u(t-1)$

From lecture $L\{u(t)\} = \frac{1}{s}$ and $L\{x(t-t_d)\} = e^{-st_d}X(s)$

$X(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s}$, for all s

This is a finite-amplitude finite-duration signal, and hence the region of convergence is the entire s plane.

At first glance, it would seem that the region of convergence should have been Re{s} > 0 because the signal is causal and the denominator goes to zero when *s* goes to zero. However, when *s* goes to zero, the numerator also goes to zero. We can use L'Hôpital's rule by letting *s* go to zero. The derivative of the numerator with respect to s is exp(-*s*) and the derivative of the denominator with respect to s is 1. The limit of *X*(*s*) as *s* goes to zero is 1.

b) $x(t) = 3e^{-3t}u(t-2)$

From $L\{e^{-at}u(t)\} = \frac{1}{s+a}$ and region of convergence is Re{*s*} > -Re{*a*}.

$x(t) = 3e^{-3t}u(t-2) = 3e^{-3(t-2+2)}u(t-2) = 3e^{-6}e^{-3(t-2)}u(t-2)$

$X(s) = \frac{3e^{-6}e^{-2s}}{s+3} = \frac{3e^{-6}}{s+3}e^{-2s}$

And the region of convergence is Re{*s*} > -3.

Writing the exp(-2*s*) separately from the rest of the expression can help highlight the term, which corresponds in the time domain to a delay by 2 seconds.

c) $x(t) = 3e^{-3(t-2)}u(t-2)$

This part is similar to part b.

$X(s) = \frac{3e^{-2s}}{s+3} = \frac{3}{s+3}e^{-2s}$

The region of convergence is Re{*s*} > -3.

d) $x(t) = 5\sin\left(\pi(t-1)\right)u(t-1)$

From $L\{\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$ and region of convergence is Re{*s*} > 0.

$X(s) = \frac{5e^{-s}\pi}{s^2+\pi^2} = \frac{5\pi}{s^2+\pi^2}e^{-s}$

And the region of convergence is Re{*s*} > 0.

Problem 2.

a)

$$x(t) = \cos\left(20\pi t\right)u(t)$$

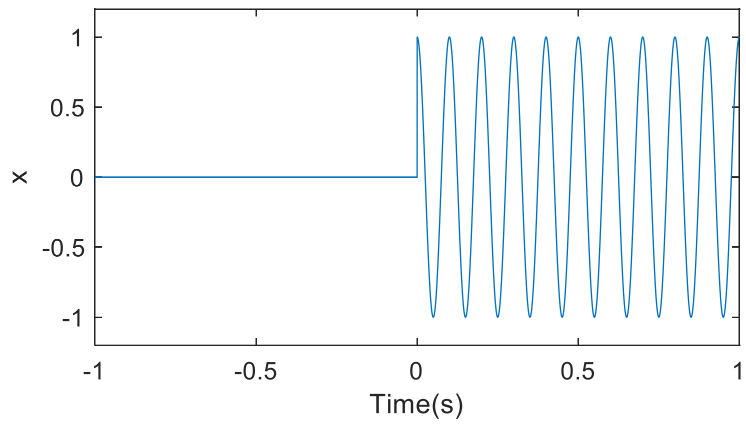
From $L\left\{\cos\left(\omega_0 t\right)u(t)\right\} = \dfrac{s}{s^2 + \omega_0^2}$ for Re{s} >0.

$$X(s) = \frac{s}{s^2 + \left(20\pi\right)^2} = \frac{s}{s^2 + 400\pi^2}$$

The region of convergence is Re{s} > 0.

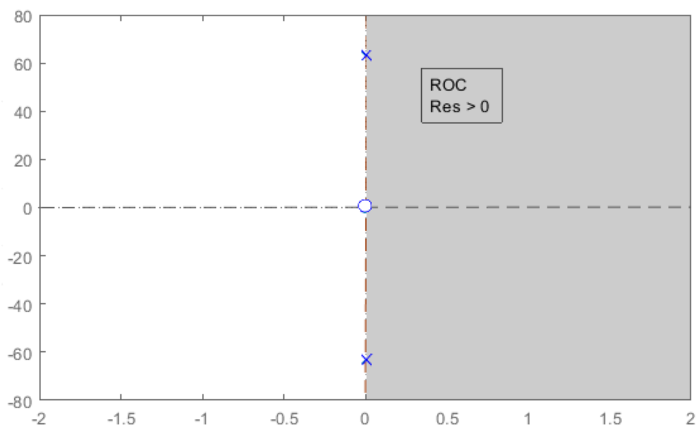
MATLAB code

```
t = -1:1/10000:1;
unitstep = zeros(size(t));
unitstep (t>= 0) = 1;
x = cos(20*pi*t).*unitstep;
plot(t,x)
xlabel('Time(s)')
ylabel('x')
```



Zeros are roots of nominator and poles are roots of denominator.

$zero: s = 0$, $\quad pole: s^2 + \left(20\pi\right)^2 = 0 \Rightarrow s_{1,2} = \pm j20\pi$ In the figure legend, Res means Re{s}.



b)

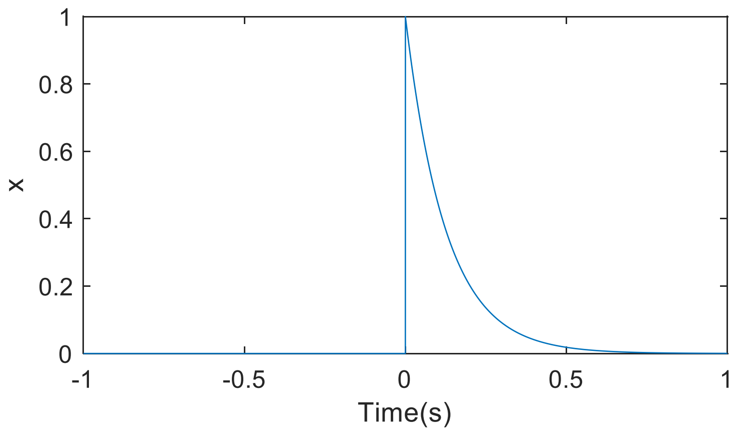
$x(t) = e^{-8t}u(t)$

From $L\left\{e^{-at}u(t)\right\} = \dfrac{1}{s+a}$ for Re{s} > -Re{a}.

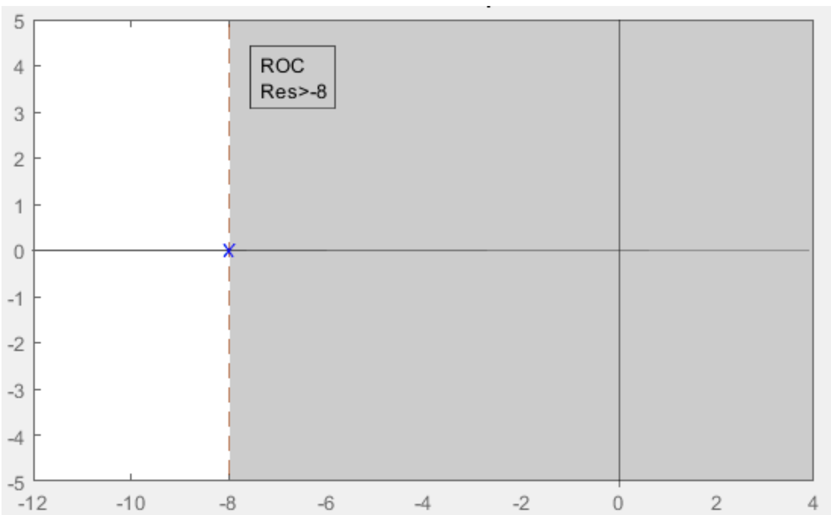
$X(s) = \dfrac{1}{s+8}$

The region of convergence is Re{s} > -8.

```
t = -1:1/10000:1;
unitstep = zeros(size(t));
unitstep (t>= 0) = 1;
x = exp(-8*t).*unitstep;
plot(t,x)
xlabel('Time(s)')
ylabel('x')
```



$pole: s+8=0 \Rightarrow s=-8$ In the figure legend, Res means Re{s}.



c)

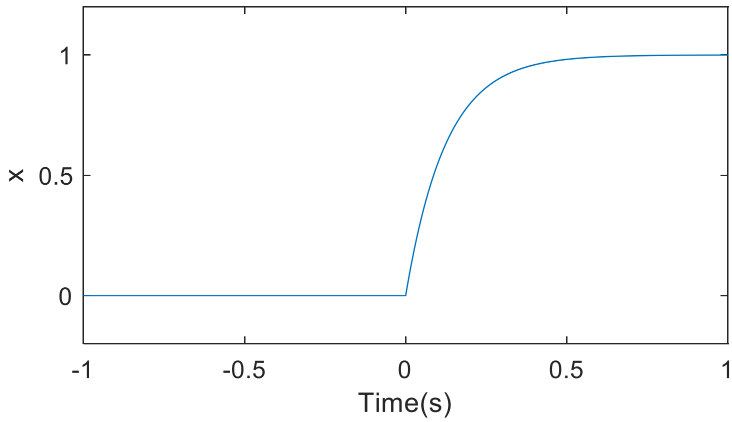
$$x(t) = (1 - e^{-8t})u(t) = u(t) - e^{-8t}u(t)$$

From $L\left\{e^{-at}u(t)\right\} = \dfrac{1}{s+a}$ for Re{s} > -Re{a}.

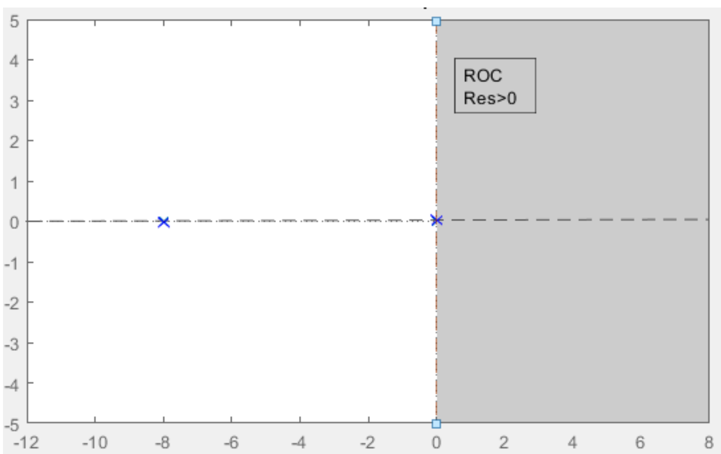
$$X(s) = \frac{1}{s} - \frac{1}{s+8} = \frac{8}{s(s+8)}$$

The region of convergence is Re{s} > 0.

```
t = -1:1/10000:1;
unitstep = zeros(size(t));
unitstep (t>= 0) = 1;
x = (1-exp(-8*t)).*unitstep;
plot(t,x)
xlabel('Time(s)')
ylabel('x')
```



$poles: s(s+8) = 0 \Rightarrow s_1 = 0, s_2 = -8$ In the figure legend, Res means Re{s}.



Problem 3.

Using the property: $L\left\{\frac{d}{dt}x(t)\right\} = sL\{x(t)\}$ for zero initial conditions, we get:

a) $sY(s) + 2Y(s) = sX(s) => Y(s)(s+2) = sX(s) => H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s+2}$

Because the system is causal, the region of convergence is Re{s} > -2.

b) Using the Laplace transform pair $L\{e^{-at}u(t)\} = \frac{1}{s+a}$ for Re{s} > -Re{a}, $L(\delta(t)) = 1$ for all s, we obtain

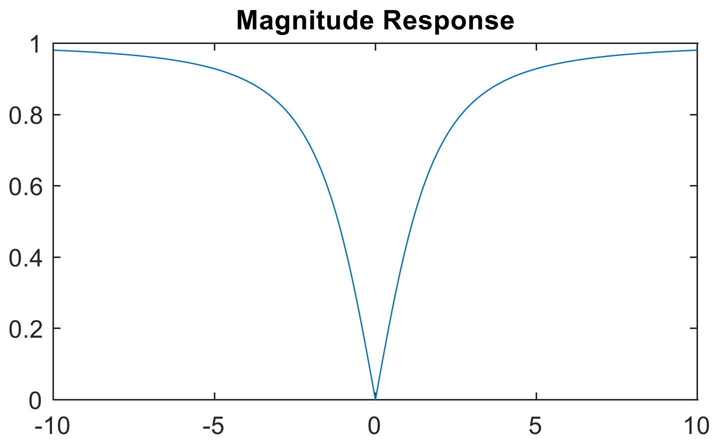
$$H(s) = \frac{s}{s+2} = \frac{s+2-2}{s+2} = 1 - \frac{2}{s+2}$$

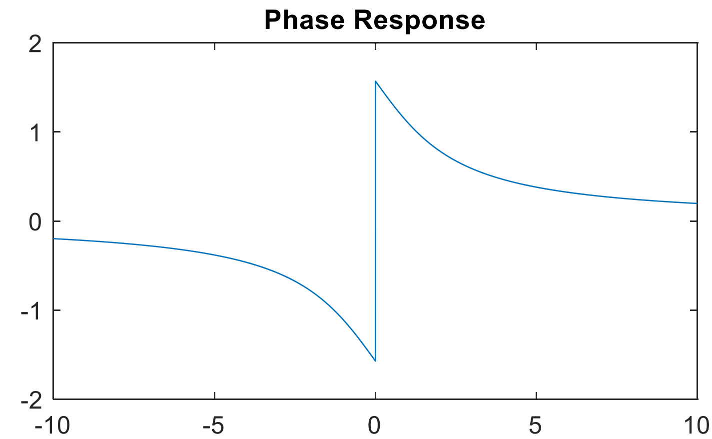
$$h(t) = \delta(t) - 2e^{-2t}u(t)$$

c) $H(j\omega) = \frac{j\omega}{j\omega+2}$ by substituting $s = j\omega$ into $H(s)$ above. This substitution is valid because the imaginary axis lies within the region of convergence of Re{s} > -2.

d)

```
w = -10:1/10000:10;
H= j*w./(j*w+2);
Hmag=abs(H) ;
Hphase=angle(H);
plot(w,Hmag)
title('Magnitude Response');
figure
plot(w,Hphase)
title('Phase Response');
```

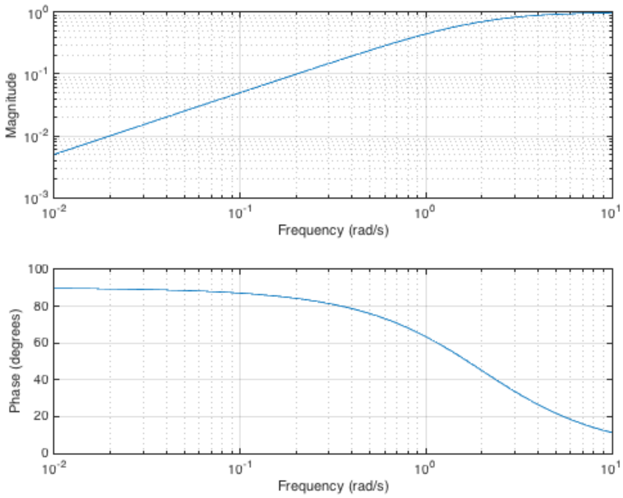




According to magnitude response, the filter notches out zero frequency. Hence, it is a notch filter. It could also be called a highpass filter, but a DC notch filter would be more descriptive and a better answer.

Here's the plot of the magnitude and phase using the `freqs` command in Matlab, which will plot the frequency responses on a log scale in frequency. The magnitude will also be on a log scale.

```
freqs( [1 0], [1 2] );
```



We see a highpass response over the frequencies plotted.

Please note that `freqs( [1], [1 2] )` would mean $\frac{1}{s+2}$ for the transfer function instead of $\frac{s}{s+2}$.

e) $x(t) = u(t) \Rightarrow X(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = \frac{1}{s}$ for Re{s} > 0. We could have also obtained the transform by using $L\{e^{-at}u(t)\} = \frac{1}{s+a}$ and substituting *a* = 0.

$$Y(s) = H(s)X(s) = \frac{1}{s+2}$$

f)

Using the Laplace transform pair $L\{e^{-at}u(t)\} = \frac{1}{s+a}$, for we get

$y(t) = e^{-2t}u(t)$

Problem 4.

a) Using the Laplace transform pair $L\{e^{-at}u(t)\} = \frac{1}{s+a}$, we get

$h(t) = e^{-2t}u(t)$

b)

$$X(s) = L\left\{7e^{-2t}u(t)\right\} = \frac{7}{s+2}, \textit{for} \operatorname{Re}\{s\} > -2$$

$$Y(s) = H(s)X(s) = \frac{7}{(s+2)^2}, \text{for} \operatorname{Re}\{s\} > -2$$

From $L\left\{t^r e^{-at}u(t)\right\} = \frac{r!}{(a+s)^{r+1}}, \text{for} \operatorname{Re}\{s\} > -\operatorname{Re}\{a\}$

$y(t) = 7\ t\ e^{-2t}\ u(t)$

c) Using convolution:

$$y(t) = h(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau$$

$$= \int_{-\infty}^{t} e^{-2\tau}u(\tau)e^{-2(t-\tau)}d\tau = e^{-2t}\int_{-\infty}^{t} u(\tau)d\tau = \begin{cases} 0, & t < 0 \\ e^{-2t}\int_{0}^{t} 1 d\tau = te^{-2t}, & t \geq 0 \end{cases}$$

$$h(t) * h(t) = t\ e^{-2t}\ u(t)$$

For a discrete-time version of this problem, please see Handout F Convolution of Exponential Sequences at
http://users.ece.utexas.edu/~bevans/courses/signals/handouts/Appendix%20F%20Convolution%20Exp%20Sequences.pdf

d)

From $L\left\{t^r e^{-at}u(t)\right\} = \dfrac{r!}{(a+s)^{r+1}}, \textit{for} \operatorname{Re}\{s\} > -\operatorname{Re}\{a\}$

$$y(t) = h(t) * h(t) \to Y(s) = L\left\{te^{-2t}u(t)\right\} = \frac{1}{(s+2)^2}, \textit{for} \operatorname{Re}\{s\} > -2$$

Poles are the roots of the denominator:

$$poles : (s+2)^2 = 0 \Rightarrow s_{1,2} = -2$$

Thus, $Y(s)$ has a double pole at s = -2.