

Solution Set for Homework #2 on Fourier Series

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1. **Prologue:** This problem asks to generate the signal in the time domain by using the signal's spectrum. The spectral and time representations give complementary views into the signal.

Solution: The spectrum indicates that the signal has frequency components of -175 Hz, -50 Hz, 0 Hz, 50 Hz and 175 Hz. The strongest frequency component is at 0 Hz because its magnitude (11) is the largest. The second strongest frequency components are at -50 Hz and 50 Hz. The weakest frequency components are at -175 Hz and 175 Hz.

Part (a): We can directly read off the spectral components to create the time domain representation of the signal

$$\begin{aligned} x(t) &= 4e^{j\pi/2}e^{-j2\pi(175)t} + 7e^{j\pi/3}e^{-j2\pi(50)t} + 11e^{j0t} + 7e^{-j\pi/3}e^{j2\pi(50)t} + 4e^{-j\pi/2}e^{j2\pi(175)t} \\ &= 11 + \frac{14e^{-j\pi/3}e^{j2\pi(50)t} + 14e^{j\pi/3}e^{-j2\pi(50)t}}{2} + \frac{8e^{-j\pi/2}e^{j2\pi(175)t} + 8e^{j\pi/2}e^{-j2\pi(175)t}}{2} \\ &= 11 + 14\cos(2\pi(50)t - \pi/3) + 8\cos(2\pi(175)t - \pi/2) \end{aligned}$$

Part (b): The signal $x(t)$ is periodic with period T , if $x(t + T) = x(t)$ for all values of t :

$$x(t) = 11 + 14\cos(2\pi(50)t - \pi/3) + 8\cos(2\pi(175)t - \pi/2)$$

$$x(t+T) = 11 + 14\cos(2\pi(50)(t+T) - \pi/3) + 8\cos(2\pi(175)(t+T) - \pi/2)$$

$$= 11 + 14\cos(2\pi(50)t + 2\pi(50)T - \pi/3) + 8\cos(2\pi(175)t + 2\pi(175)T - \pi/2)$$

In order for $x(t + T) = x(t)$ for all values of t , we can equate the first terms in $x(t + T)$ and $x(t)$, equate the second terms in $x(t + T)$ and $x(t)$, and equate the third terms in $x(t + T)$ and $x(t)$, and find values of T that work for all three terms.

The first terms are already equal to each other (11). The second terms are equal when $2\pi(50)T = 2\pi m$ where m is an integer, i.e. when T is $1/50, 2/50, 3/50, 4/50$, etc. The third terms are equal when $2\pi(175)T = 2\pi k$ where k is an integer; i.e., when T is $1/175, 2/175, 3/175, 4/175$, etc. The common values for the period T in seconds among all three terms are $1/25, 2/25, 3/25, 4/25$, etc., which means that the fundamental period is 0.04 s, and the fundamental frequency is 25 Hz.

Alternately, the fundamental frequency can be found from the spectrum as the greatest common divisor (gcd) of (50, 175): $f_0 = \text{gcd}(50, 175) = 25$ Hz and $T_0 = 1/f_0 = 1/25 = 0.04$ s

Part (c): $\cos(\Theta) = \text{Re}\{e^{j\Theta}\}$. From the inverse Euler's formula, $\cos(\Theta) = \frac{1}{2}e^{j\Theta} + \frac{1}{2}e^{-j\Theta}$.

Therefore sinusoids should be shown by both frequencies.

Epilog: Sometimes, it's easier to answer a question in the time domain, and sometimes, using the frequency domain is easier. In part (b), it was much easier to find the fundamental frequency from the spectrum than the time-domain representation. And, sometimes, a time-frequency representation is the right tool for the problem, as we'll see in problem 3.

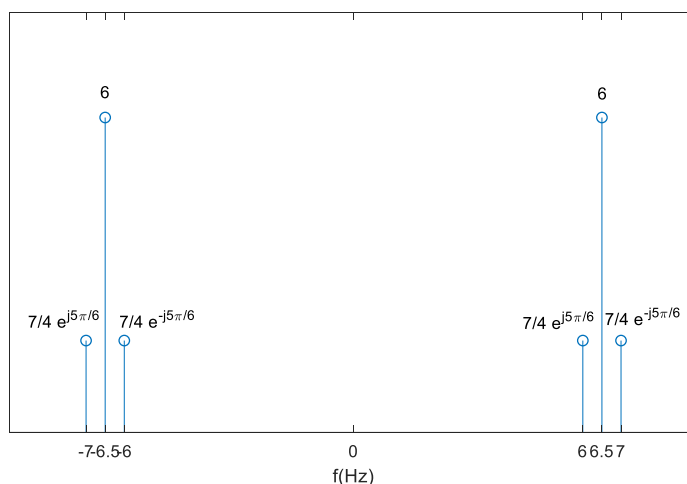
2. Prologue: Amplitude modulation is a powerful idea. It is used to transmit a message by shifting the frequency content in the message to a higher frequency to make it easier to transmit. Consider the transmission of an audio signal 30 miles (48 km), e.g. from Austin, TX, to San Marcos, TX. Cranking up the volume on the audio signal using the largest bank of speakers possible in Austin would not make the audio signal audible in San Marcos. However, we could use amplitude modulation to shift the audio signal by 1300 kHz and transmit it as an electromagnetic wave over the air, which would be using an AM Radio Station at 1300 kHz. A receiver in San Marcos would receive the AM transmission and shift the frequencies from 1300 kHz to 0 Hz, which is known as demodulation. Long-range propagation is possible in AM radio bands. The idea of amplitude modulation is used in light communications over optical fiber (a.k.a. optical communications) as well as in cellular and Wi-Fi communications. Amplitude modulation can also create audio effects. This particular problem looks at amplitude modulation used in AM radio, which adds a large DC offset to the message signal before the frequency content is shifted.

Part (a): The message signal is $12 + 7 \sin(\pi t - \pi/3)$ which has frequency components of -0.5 Hz, 0 and 0.5 Hz. The modulating (carrier) signal is $\cos(13 \pi t)$, whose frequency is at 6.5 Hz.

$$\begin{aligned}
 x(t) &= [12 + 7 \sin(\pi t - \pi/3)] \cos(13\pi t) = [12 + 7 \cos(\pi t - \pi/3 - \pi/2)] \cos(13\pi t) \\
 &= [12 + \frac{7}{2}(e^{-j5\pi/6} e^{j\pi t} + e^{j5\pi/6} e^{-j\pi t})] \frac{e^{j13\pi t} + e^{-j13\pi t}}{2} \\
 &= 6e^{j13\pi t} + 6e^{-j13\pi t} + \frac{7}{4}[e^{-j5\pi/6} e^{j14\pi t} + e^{-j5\pi/6} e^{-j12\pi t} + e^{j5\pi/6} e^{j12\pi t} + e^{j5\pi/6} e^{-j14\pi t}] \\
 &= (6e^{j13\pi t} + 6e^{-j13\pi t}) + (\frac{7}{4} e^{-j5\pi/6} e^{j14\pi t} + \frac{7}{4} e^{j5\pi/6} e^{-j14\pi t}) + (\frac{7}{4} e^{j5\pi/6} e^{j12\pi t} + \frac{7}{4} e^{-j5\pi/6} e^{-j12\pi t}) \\
 &= \frac{7}{2} \cos(12\pi t + \frac{5\pi}{6}) + 12 \cos(13\pi t) + \frac{7}{2} \cos(14\pi t - \frac{5\pi}{6})
 \end{aligned}$$

$A_1 = 7/2 = 3.5$	$\omega_1 = 12\pi$ (rad/s)	$\phi_1 = 5\pi/6$ (rad)
$A_2 = 12$	$\omega_2 = 13\pi$ (rad/s)	$\phi_2 = 0$ (rad)
$A_3 = 7/2 = 3.5$	$\omega_3 = 14\pi$ (rad/s)	$\phi_3 = -5\pi/6$ (rad)

Part (b): Note that the message signal has been shifted to the right by 6.5 Hz. A similar shift in frequency also occurs to the left, which makes sense because $x(t)$ is real-valued.



3. **Prologue:** In a chirp signal, as mentioned in Section 3-8 of *Signal Processing First*, the principal frequency increases (or decreases) with time.

In active sonar systems, the transmitter plays out a “ping” as sound in form of a chirp and then receives sound. The time elapsed between the transmission and reception of the chirp indicates the round trip time experienced by the signal after bouncing off an object in the water and returning to the receiver. By receiving sounds in different directions using multiple microphones, the sonar can build a map of the objects in the water.

This kind of ranging approach can also be used for positioning and navigation. Bats use chirps for echolocation. Pipistrelle bats uses chirps that sweep down from 70 to 45 kHz. <https://www.wildlife-sound.org/resources/equipment/2-uncategorised/233-recordings-of-ultrasonic-vocalisations-of-bats>

When one measures the response of a system to different frequencies, a time-consuming approach is to input a single sinusoid, measure the output, and repeat using many different frequencies. Instead, inputting a chirp can allow the measurement to be performed in one take.

4G cellular communication systems periodically send a Zadoff-Chu chirp sequence to help measure the distortion in the electromagnetic propagation from transmitter to receiver.

Part (a): MATLAB code to generate the chirp signal:

```
time = 10;           % length of time in seconds
f0 = 20;            % specify starting principal frequency
fstep = 420;       % specify frequency slope
fs = 44100;        % sampling rate
Ts = 1/fs;         % sampling time: time interval between samples
t = 0 : Ts : time; % create a time vector
theta = 2*pi*(f0+0.5*fstep*t).*t;
y = cos(theta);    % create chirp waveform
```

Part b: The chirp signal can be played as an audio signal in MATLAB:

```
sound(y, fs);      % play back chirp signal
```

The chirp linearly sweeps frequencies from 20 to 4220 Hz, and sounds like a note increasing in “pitch” over time. The sweep spans 8 octaves of A notes on a Western scale: 27.5, 55, 110, 220, 440, 880, 1760, 3520 Hz. See https://en.wikipedia.org/wiki/Piano_key_frequencies.

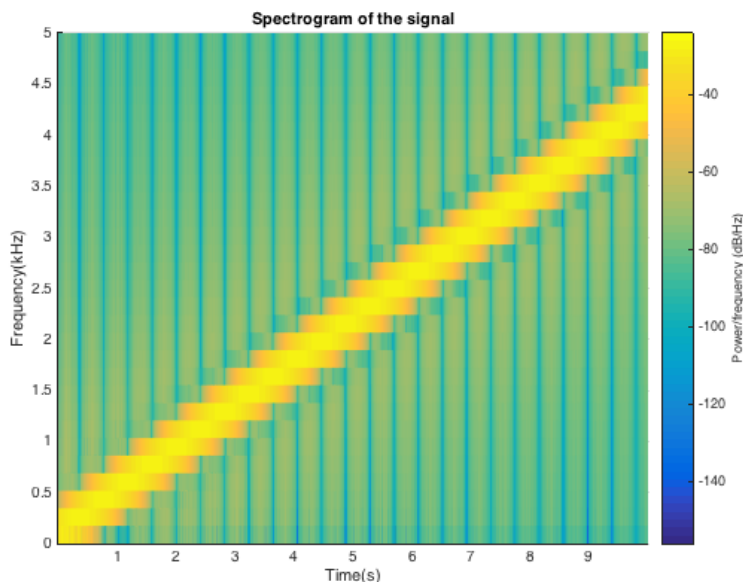
Part (c): We can visualize the variation of the principal frequency over time by using a time-frequency representation. The time-frequency representation below is called a spectrogram.

The spectrogram for the chirp signal (next page) shows the principal frequency changing over time with a linear slope. At the beginning, the principal frequency is at 20 Hz and increases linearly to 4220 Hz. The principal frequency has the highest magnitude at every instant of time throughout the entire duration of the chirp signal, which is shown in yellow.

As described on lecture slides 4-8 and 4-9, a spectrogram takes the first N_{win} samples of the signal, weights the values (using a rectangular pulse by default), computes the Fourier series coefficients, and plots the magnitude of the Fourier series coefficients vertically. The spectrogram then shifts the time signal to the right and repeats the previous steps using a block of the next N_{win} samples. The frequency resolution of the spectrogram is f_s / N_{win} .

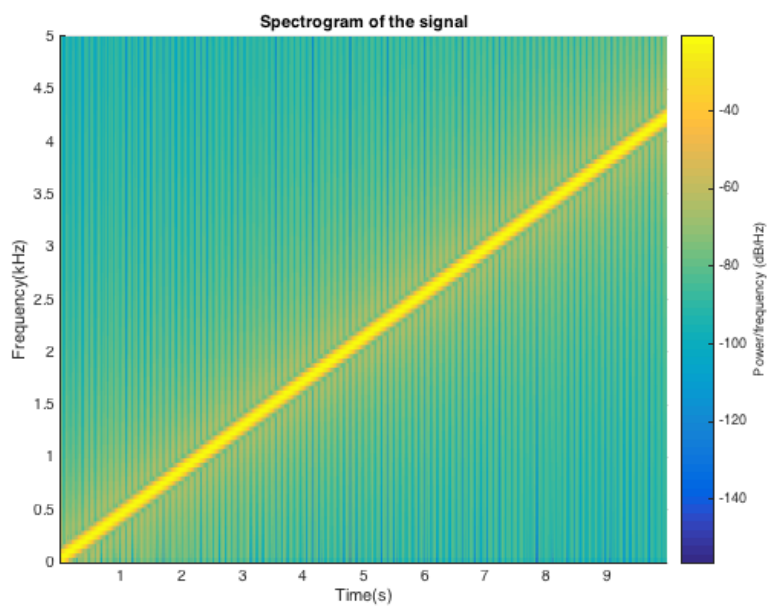
In the code below, a Hamming window is used to weight the values in each block of samples.

```
figure;  
spectrogram(y, hamming(256), 128, 256, fs, 'yaxis')  
ylim([0,5]);  
title('Spectrogram of the signal');  
ylabel('Frequency(kHz)');  
xlabel('Time(s)');
```



Although not asked, we can increase the frequency resolution by increasing the block size:

```
figure;  
spectrogram(y, hamming(1024), 512, 1024, fs, 'yaxis')  
ylim([0,5]);  
title('Spectrogram of the signal');  
ylabel('Frequency(kHz)');  
xlabel('Time(s)');
```



4. **Prologue:** This problem explores various audio effects by applying nonlinear operations to a sinusoidal signal. It also explores what happens to the frequency component of a sinusoidal signal due to nonlinear operations. This is a common approach in analyzing how a nonlinear system (e.g. a diode or transistor) responds to different frequencies. Among the nonlinear operations, parts (a), (b) and (c) can be used in amplitude demodulation (see the Prologue for Problem 2). The nonlinear operation in part (d) is a type of frequency modulation.

Solution: MATLAB code for generating and playing $x(t)$ and output signals for parts (a)-(d):

```
f0 = 440;
fs = 8000;
Ts = 1/fs;
t = 0:Ts:5;
x = cos(2*pi*f0*t);
y_a = abs(x);
y_b = x.^2;
y_c = x.^4;
y_d = cos(pi*x);

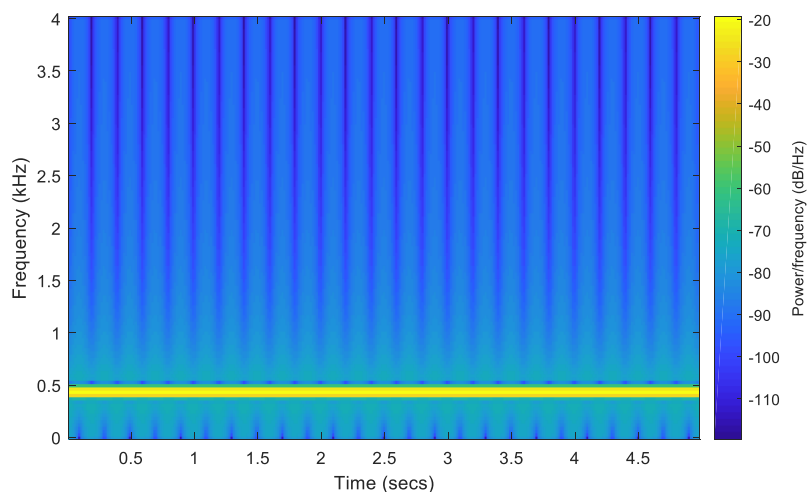
figure(1)
spectrogram(x, hamming(256), 128, 256, fs, 'yaxis')
sound(x, fs)
pause(8);

figure(2)
spectrogram(y_a, hamming(256), 128, 256, fs, 'yaxis')
soundsc(y_a, fs)
pause(8);

figure(3)
spectrogram(y_b, hamming(256), 128, 256, fs, 'yaxis')
soundsc(y_b, fs)
pause(8);

figure(4)
spectrogram(y_c, hamming(256), 128, 256, fs, 'yaxis')
soundsc(y_c, fs)
pause(8);

figure(5)
spectrogram(y_d, hamming(256), 128, 256, fs, 'yaxis')
soundsc(y_d, fs)
```



$x(t)$ only contains the frequencies ± 440 Hz, and the spectrogram shows this.

Part (a): The fundamental period of $y(t)$ is different from fundamental period of $x(t)$.
 In order to be periodic $y(t) = y(t+T_0)$ and T_0 is the fundamental period where $T_0 = 1/880$ (s)

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi k}{T_0}t}$$

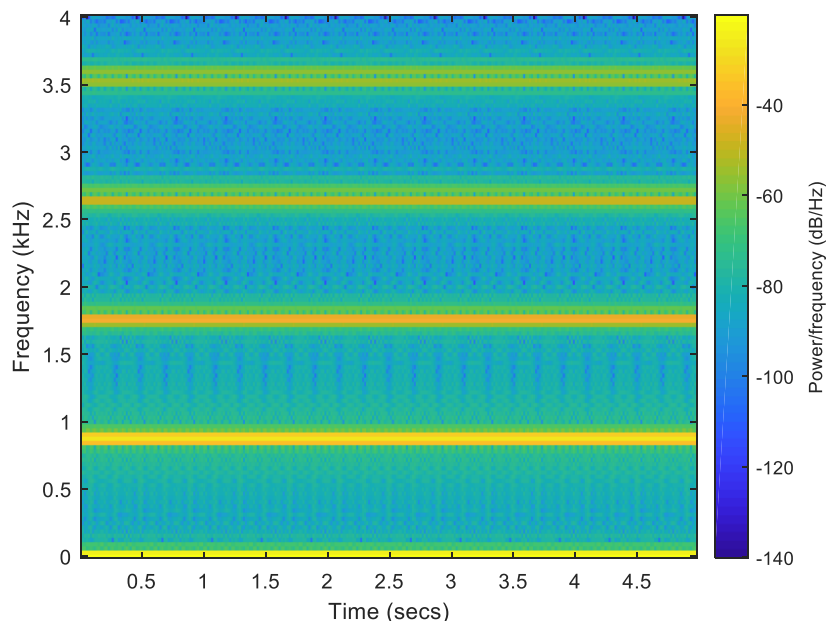
$$a_k = \frac{1}{T_0} \int_0^{T_0} y(t) e^{-j\frac{2\pi k}{T_0}t} dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t) e^{-j\frac{2\pi k}{T_0}t} dt$$

$$a_0 = 880 \int_{-\frac{1}{1760}}^{\frac{1}{1760}} \cos(2\pi(440)t) dt = \frac{880}{2\pi(440)} (\sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2})) = \frac{2}{\pi}$$

And

$$\begin{aligned} a_k &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t) e^{-j2\pi k f_0 t} dt = 880 \int_{-\frac{1}{1760}}^{\frac{1}{1760}} |\cos(2\pi(440)t)| e^{-j2\pi k(880)t} dt = 880 \int_{-\frac{1}{1760}}^{\frac{1}{1760}} \cos(2\pi(440)t) e^{-j2\pi k(880)t} dt = \\ &= 880 \int_{-\frac{1}{1760}}^{\frac{1}{1760}} \frac{e^{j2\pi(440)t} + e^{-j2\pi(440)t}}{2} e^{-j2\pi k(880)t} dt = 880 \int_{-\frac{1}{1760}}^{\frac{1}{1760}} \frac{e^{-j2\pi(880k-440)t} + e^{-j2\pi(880k+440)t}}{2} dt \\ &= \frac{880}{2} \left[-\frac{1}{j2\pi(880k-440)} (e^{-j2\pi(0.5k-0.25)} - e^{j2\pi(0.5k-0.25)}) - \frac{1}{j2\pi(880k+440)} (e^{-j2\pi(0.5k+0.25)} - e^{j2\pi(0.5k+0.25)}) \right] \\ &= e^{-j\pi k} \left[-\frac{1}{2\pi(2k-1)} + \frac{1}{2\pi(2k+1)} \right] + e^{j\pi k} \left[-\frac{1}{2\pi(2k-1)} + \frac{1}{2\pi(2k+1)} \right] = \frac{2 \cos(\pi k)}{\pi(1-4k^2)} \end{aligned}$$

So, $y(t)$ contains an infinite number of even harmonics of 440 Hz, which correspond to frequencies of $\pm 880k$ where k is an integer value. As expected, we can hear a higher frequency sound. The sound feels as higher number of frequencies are present (less 'thin' than a pure cosine). The Sampling Theorem says to choose the sampling rate so that $f_s > 2 f_{\max}$. After dividing both sides by 2, $f_{\max} < \frac{1}{2} f_s$. Sampling at 8 kHz will only capture harmonic frequencies below 4 kHz.



Part (b):

$$y(t) = \cos^2(2\pi 440t)$$

$$y(t) = \frac{1}{2} + \frac{1}{2} \cos(2\pi 880t)$$

(Using the trig. identity $\cos(2\theta) = 2\cos^2(\theta) - 1$)

$y(t)$ contains 0 Hz and ± 880 Hz, which are even harmonics of 440 Hz up to the 2nd harmonic.

Alternatively,

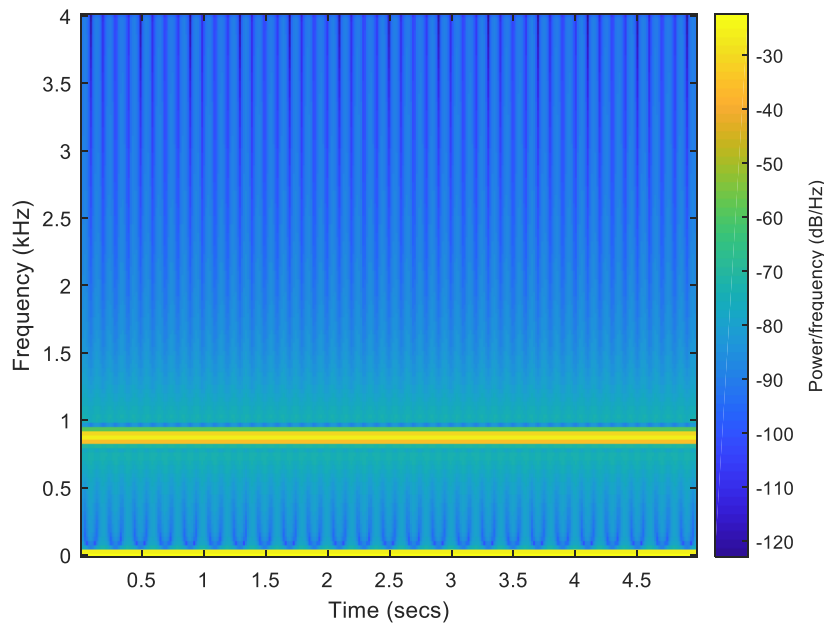
$$y(t) = \cos^2(2\pi 440t)$$

$$y(t) = (e^{j2\pi 440t} + e^{-j2\pi 440t})^2 / 4$$

$$= \frac{e^{j2\pi 880t} + e^{-j2\pi 880t} + 2}{4} = \frac{1}{2} + \frac{\cos(2\pi 880t)}{2}$$

(Using the binomial expansion for $(a+b)^2 = a^2 + b^2 + 2ab$)

It sounds 'thin', which is because just one frequency is present in the waveform, while in part a, higher frequencies could be heard.



Part (c):

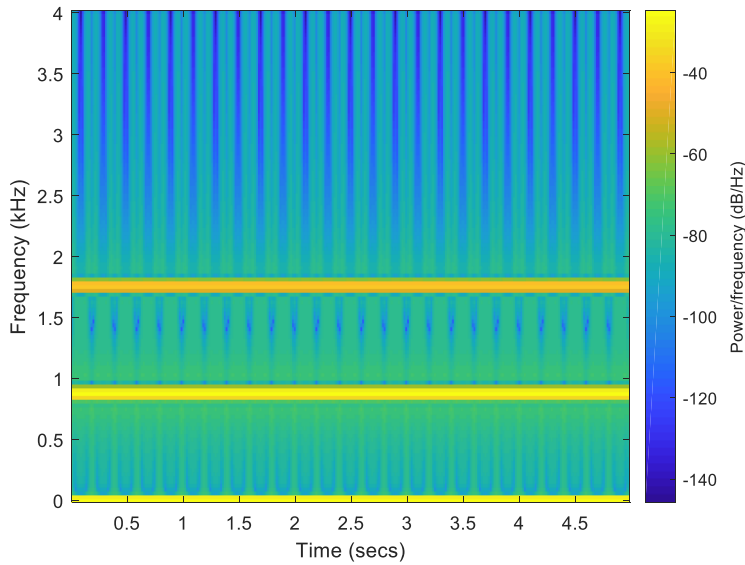
$$y(t) = x^4(t) = \cos^4(2\pi f_0 t) = \left(\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \right)^4 = \frac{e^{j8\pi f_0 t} + e^{-j8\pi f_0 t} + 4e^{j4\pi f_0 t} + 4e^{-j4\pi f_0 t} + 6}{16}$$

$$= 0.375 + 0.5 \cos(4\pi f_0 t) + 0.125 \cos(8\pi f_0 t) = 0.375 + 0.5 \cos(2\pi(880)t) + 0.125 \cos(2\pi(1760)t)$$

(Using the binomial expansion for $(a + b)^4 = a^4 + b^4 + 4 a^3 b + 4 a b^3 + 6a^2 b^2$)

So, $y(t)$ contains the frequencies 0 Hz, ± 880 Hz and ± 1760 Hz, which are the even harmonics of 440 Hz up to the 4th harmonic.

This sounds as if it has a frequency between the signals in parts (a) & (b). The sound is more 'thin' than part (a), while it is not pure as audio in part b, and the other frequency (± 1760 Hz) can be heard.



Part (d): Using Taylor series expansion:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - L$$

$$\cos(\pi \cos(2\pi f_0 t)) = 1 - \frac{(\cos(2\pi f_0 t))^2}{2!} + \frac{(\cos(2\pi f_0 t))^4}{4!} - \frac{(\cos(2\pi f_0 t))^6}{6!} + \frac{(\cos(2\pi f_0 t))^8}{8!} - L$$

As shown in the above formula, $y(t)$ includes $(\cos(\pi \cos(2\pi f_0 t)))^2$, and as shown in part (b) and (c), $y(t) = \cos^n(2\pi f_0 t)$ will have odd harmonics up to the n th harmonic if n is odd, and even harmonics up to the n th harmonic if n is even including a zero-frequency component.

Compared to parts (b) and (c), the audio signal has additional harmonics.

