## *Solution Set for Homework #3 on Fourier Series and Sampling*

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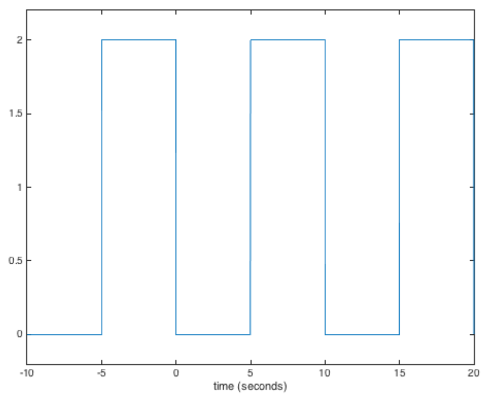
**1. Prolog:** This problem relates to square waves, their Fourier series, and how adding a DC value affects Fourier series coefficients.

**Part (a):** This signal is periodic with a fundamental period of $T_o$ = 10 s, and hence the fundamental frequency is $f_0 = 1 / T_0$ = 0.1 Hz. From Section 3.6.1 of *Signal Processing First*, the square wave has an infinite number of harmonic frequencies, and their strength decays inversely proportionally to the harmonic index *k*. For plotting the time domain signal in MATLAB, we'll want to pick a sampling rate $f_s$ that is a multiple of $f_0$ and that can capture the sudden transitions in amplitude at -5s, 0s, 5s, 10s, 15s, and 20s. We'll need $f_s$ to be a large multiple of 2 $f_0$. For the plot below, we've chosen $f_s$ to be 10 kHz.




```
T0 = 10;
f0 = 1/T0;
fs = 10^6 * f0;
Ts = 1 / fs;
t = -10 : Ts : 20;
% mod(t, T0) puts t in the fundamental period
% comparisons return 1 if true and 0 if false
x = 2 * ( mod(t, T0) >= 5 );
plot(t, x);
ylim( [-0.2 2.2] );
xlabel( 'time (seconds)');
```

**Part (b):**

To calculate $a_o$, the DC value, or signal average over a period should be calculated.

$$a_o = \frac{1}{10}(5\times 2) = 1$$

**Part (c):**

$$a_1 = \frac{1}{10}\int_5^{10} 2e^{-j\frac{2\pi}{10}t}dt = \frac{1}{10}\times\frac{1}{-j\frac{2\pi}{10}}2e^{-j\frac{2\pi}{10}t}\Bigg|_5^{10} = -\frac{1}{j\pi}(e^{-j2\pi} - e^{-j\pi}) = j\frac{2}{\pi} = \frac{2}{\pi}e^{j\pi/2}$$

**Part (d):**

By adding a constant value to a signal, only the DC value of that signal changes. Therefore, all Fourier series coefficient, except $a_0$ remain unchanged.

$b_0 = a_0 + 1 = 2$

$$b_1 = a_1 = \frac{2}{\pi} e^{j\pi/2}$$

2. **Prolog:** This problem concerns continuous-time sinusoids and their fundamental periods, and discrete-time sinusoids and their fundamental periods.

$$x(t) = 7\sin(11\pi t) = 7\cos(11\pi t - \frac{\pi}{2})$$

In the continuous-time domain, the fundamental period is (2/11) seconds:

$$\omega_0 = 11\pi \text{ rad/s}$$

$$f_0 = \frac{11\pi}{2\pi} = 5.5 \text{ Hz}$$

$$T_0 = \frac{2}{11} \text{ s}$$

$$\phi = -\frac{\pi}{2} \text{ rad}$$

**Part (a):**

$$\hat{\omega}_0 = 2\pi \frac{f_0}{f_s} = 2\pi \frac{5.5 \text{ Hz}}{10 \text{ Hz}} = \frac{11}{10}\pi \text{ rad/sample}$$

from sampling at $f_s$ = 10 Hz, which gives

$$x[n] = 7\cos\left(\frac{11\pi}{10} n - \frac{\pi}{2}\right)$$

This signal is under-sampled, because $f_0 > f_s/2$. The following shows aliasing due to folding that is caused by the under-sampling:

$$x[n] = 7\cos\left(\frac{11\pi}{10} n - \frac{\pi}{2}\right) = 7\cos\left(\frac{11\pi}{10} n - 2\pi n - \frac{\pi}{2}\right) = 7\cos\left(-\frac{9\pi}{10} n - \frac{\pi}{2}\right) = 7\cos\left(\frac{9\pi}{10} n + \frac{\pi}{2}\right)$$

$A = 7$ , $\varphi = \pi/2$ rad.

**Part (b):**

$$\hat{\omega} = 2\pi \frac{5.5 \text{ Hz}}{5 \text{ Hz}} = \frac{11}{5}\pi \text{ rad/sample}$$

from sampling at $f_s$ = 5 Hz, which gives

$$x[n] = 7\cos\left(\frac{11\pi}{5} n - \frac{\pi}{2}\right)$$

This signal is under-sampled, because $f_0 > f_s/2$. The following equation shows the effect of aliasing (but not related to folding) caused by the under-sampling:

$$x[n] = 7\cos\left(\frac{11\pi}{5}n - \frac{\pi}{2}\right) = 7\cos\left(\frac{11\pi}{5}n - 2\pi n - \frac{\pi}{2}\right) = 7\cos\left(\frac{\pi}{5}n - \frac{\pi}{2}\right)$$

$A = 7$, $\varphi = -\pi/2$ rad.

**Part (c):**

$$\hat{\omega}_0 = 2\pi \frac{5.5 \text{ Hz}}{15 \text{ Hz}} = \frac{11}{15}\pi \text{ rad/sample}$$

This signal is 15/11 times over sampled, because $f_0 < f_s/2$.

$$x[n] = 7\cos\left(\frac{11\pi}{15}n - \frac{\pi}{2}\right)$$

$A = 7$, $\varphi = -\pi/2$ rad

**Part (d):**

As shown at the beginning of this problem's solution:

$$f_0 = \frac{11\pi}{2\pi} = 5.5 \text{ Hz}$$

$$T_0 = \frac{2}{11} \text{ s}$$

According to the hint that is provided for this solution, which comes from Handout D on Discrete-Time Periodicity, $x[n]$ is periodic with discrete-time period of $N_0$ if $x[n] = x[n+N_0]$:

$$x[n+N_0] = 7\cos\left(\frac{11\pi}{15}(n+N_0) - \frac{\pi}{2}\right) = 7\cos\left(2\pi\frac{11}{30}n + 2\pi\frac{11}{30}N_0 - \frac{\pi}{2}\right) = 7\cos\left(2\pi\frac{11}{30}n - \frac{\pi}{2}\right)$$

Because 11 and 30 are relatively prime, the smallest possible positive integer for $N_0$ is 30 samples. Therefore, this discrete-time signal $x[n]$ is periodic with a fundamental period of 30 samples. Those 30 samples contain 11 continuous-time periods, which corresponds to 2.67 samples in each continuous-time period.

Although not required, here's a plot of $x(t)$ and $x[n]$ for part (d):

```
fs = 15;
Ts = 1/fs;
wHat = 2*pi*f0/fs;
N0 = 30;
n = 0 : N0;
yofn = cos(wHat*n);
t = 0 : 0.01 : N0;
yoft = cos(wHat*t);
figure;
stem(n, yofn);
hold;
plot(t, yoft);
```