Solution Set for Homework #4 on Finite Impulse Response (FIR) Filter

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Problem 1:

Solution:

Part a:

Using a binomial expansion.

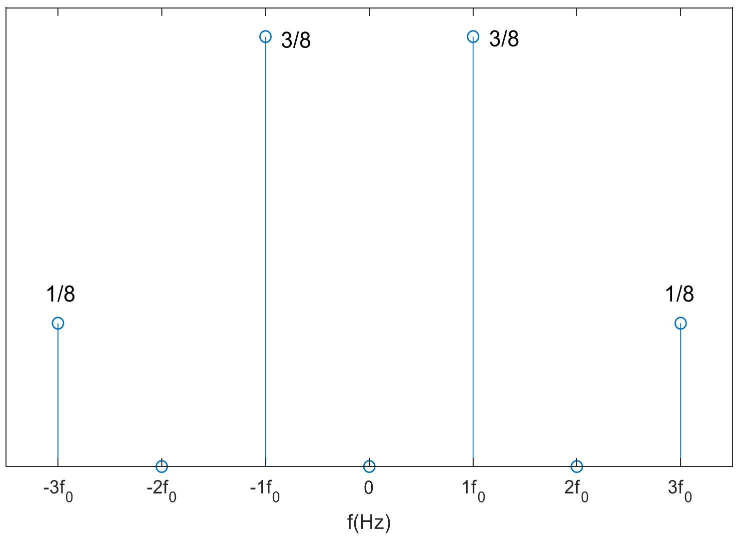
i. $$(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2) = a^3+3a^2b+3ab^2+b^3$$

ii. $$\cos(2\pi f_0 t) = \left(\frac{e^{j2\pi f_0 t}+e^{-j2\pi f_0 t}}{2}\right)$$

$$\cos^3(2\pi f_0 t) = \left(\frac{e^{j2\pi f_0 t}+e^{-j2\pi f_0 t}}{2}\right)^3 = \frac{1}{8}\left(e^{j2\pi(3f_0)t}+3e^{j2\pi f_0 t}+3e^{-j2\pi f_0 t}+e^{-j2\pi(3f_0)t}\right)$$

$$x(t) = \frac{1}{8}e^{j2\pi(3f_0)t}+\frac{3}{8}e^{j2\pi f_0 t}+\frac{3}{8}e^{-j2\pi f_0 t}+\frac{1}{8}e^{-j2\pi(3f_0)t}$$

$$\boldsymbol{a_3 = a_{-3} = \frac{1}{8} \text{ and } a_2 = a_{-2} = 0 \text{ and } a_1 = a_{-1} = \frac{3}{8} \text{ and } a_0 = 0}$$



$$\cos^3(2\pi f_0 t) = \frac{3}{4}\cos(2\pi f_0 t)+\frac{1}{4}\cos(2\pi(3f_0)t)$$

Part b:

i. $$a_0 = \frac{1}{2}\int_{-1}^{0} e^{t}dt + \frac{1}{2}\int_{0}^{1} e^{-t}dt = \frac{1}{2}\left[e^{t}\right]_{-1}^{0} + \frac{1}{2}\left[-e^{-t}\right]_{t=0}^{1} = \frac{1}{2}\left(1-e^{-1}\right) - \frac{1}{2}\left(e^{-1}-1\right) = 1-e^{-1}$$

$$a_0 = 1-e^{-1} = 0.6321$$

ii. $$a_0 = \frac{1}{T_0}\int_{0}^{T_0} \cos\left(2\pi f_0 t\right)dt = \frac{1}{2\pi f_0 T_0}\left[\sin\left(2\pi f_0 t\right)\right]_{t=0}^{T_0} = \frac{1}{2\pi}\left(0-0\right) = 0$$

Part c:

i.

We'll derive the formula by using the result from part ii below.

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} a^n = \lim_{N\to\infty} \frac{1-a^N}{1-a} = \frac{1}{1-a} \text{ if } |a| < 1$$

because

$$\lim_{N\to\infty} a^N = \begin{cases} 0 & \text{if } |a| < 1 \\ \infty & \text{otherwise} \end{cases}$$

ii.

Solution: We'll derive a closed-form answer. Let's start with a slightly different indexing for n:

$$\sum_{n=1}^{N} a^n = a + a^2 + \cdots + a^{N-1} + a^N$$

We'll reorder the addition of the terms to go from highest exponent to lowest:

$$\sum_{n=1}^{N} a^n = a^N + a^{N-1} + \cdots + a^2 + a = a\,(a^{N-1} + a^{N-2} + \cdots + a + 1)$$

The terms in parenthesis are from the result of dividing a^N-1 by a-1. We'll compute the polynomial division using long division:

			a^{N-1}	a^{N-2}	…	a	1
a-1		a^N	0	0	…	0	-1
	-	a^N	$-a^{N-1}$				
		0	a^{N-1}				
	-		a^{N-1}	$-a^{N-2}$			
			0	a^{N-2}	…		
					a^2		
	-				a^2	$-a$	
					0	a	-1
	-					a	-1
							0

$$\sum_{n=1}^{N} a^n = a\,(a^{N-1} + a^{N-2} + \cdots + a + 1) = a\left(\frac{a^N-1}{a-1}\right) = a\left(\frac{1-a^N}{1-a}\right)$$

We can connect this summation with the form in the question:

$$\sum_{n=0}^{N-1} a^n = \frac{1}{a}\sum_{n=1}^{N} a^n = \frac{1-a^N}{1-a}$$

I based the above derivation on the content at

https://www.purplemath.com/modules/series7.htm

Second solution:

Let assume:

$$S = \sum_{n=0}^{N-1} a^n$$

By multiplying "a" to both sides of this equation:

$$aS = a\sum_{n=0}^{N-1} a^n = \sum_{n=0}^{N-1} a^{n+1} = \sum_{n=1}^{N} a^n$$

$$a^0 + aS = a^0 + \sum_{n=1}^{N} a^n = \sum_{n=0}^{N} a^n$$

$$\sum_{n=0}^{N} a^n = a^N + \sum_{n=0}^{N-1} a^n = a^N + S$$

$$a^N + S = a^0 + aS$$

$$S(1-a) = a^0 - a^N = 1 - a^N$$

$$S = \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

Problem 2:

Solution:

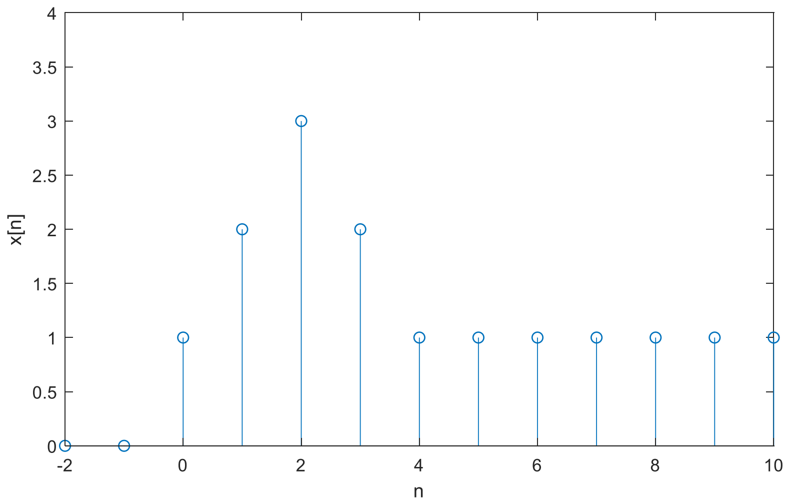
Part a:

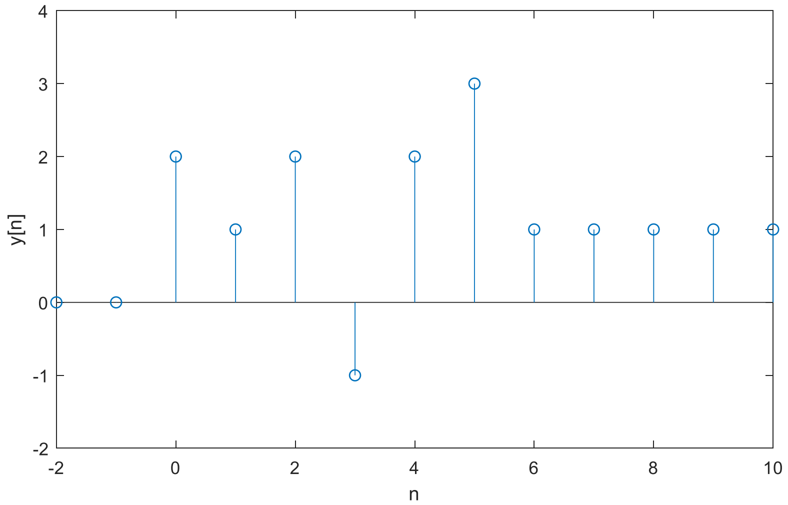
$$y[n] = 2x[n] - 3x[n-1] + 2x[n-2]$$

The values for x[n] and y[n] are given in the following table.

n	**< 0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
x[n]	0	1	2	3	2	1	1	1	1	1	1	1
x[n-1]	0	0	1	2	3	2	1	1	1	1	1	1
x[n-2]	0	0	0	1	2	3	2	1	1	1	1	1
y[n]	0	2	1	2	-1	2	3	1	1	1	1	1

Part b:





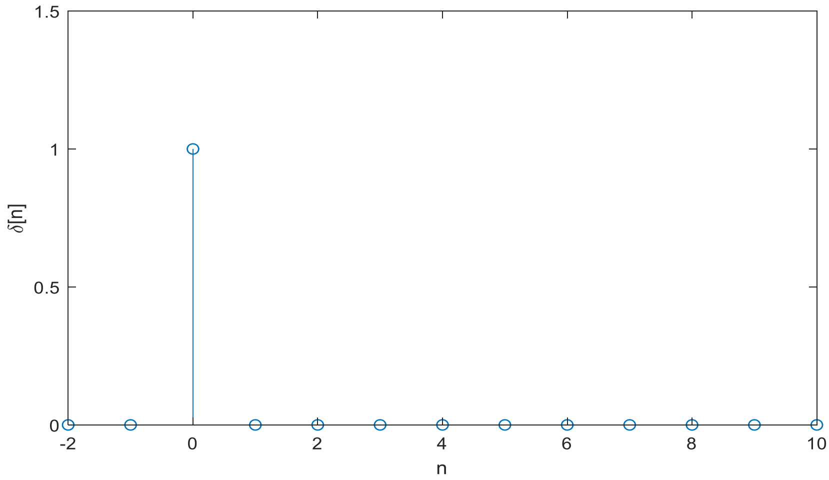
Part c:

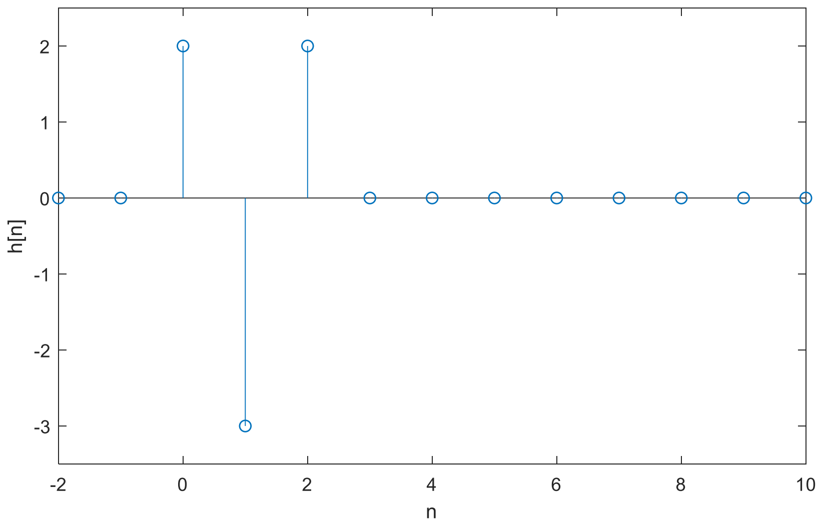
$$\delta[n] = \begin{cases} 1 & for \;\; n = 0 \\ 0 & for\, all \;\; n \neq 0 \end{cases}$$

$$h[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-2]$$

n	**< 0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
δ[n]	0	1	0	0	0	0	0	0	0	0	0	0
δ[n-1]	0	0	1	0	0	0	0	0	0	0	0	0
δ[n-2]	0	0	0	1	0	0	0	0	0	0	0	0
h[n]	0	2	-3	2	0	0	0	0	0	0	0	0

Part d:





Problem 3:

Prologue: This problem introduces the convolution sum, and asks you to calculate it for a simple finite impulse response filter (*L*-point averaging filter) given an infinitely long input signal. The unit step signal models a physical action such as turning on a switch and leaving it on indefinitely. In discrete time, the unit step function $u[n]$ is zero in amplitude for $n < 0$, and one in amplitude for $n \geq 0$.

Part a:

The MATLAB function `stepfun(n, n0)` implements $u[n-n_0]$ and is plotted on the right:

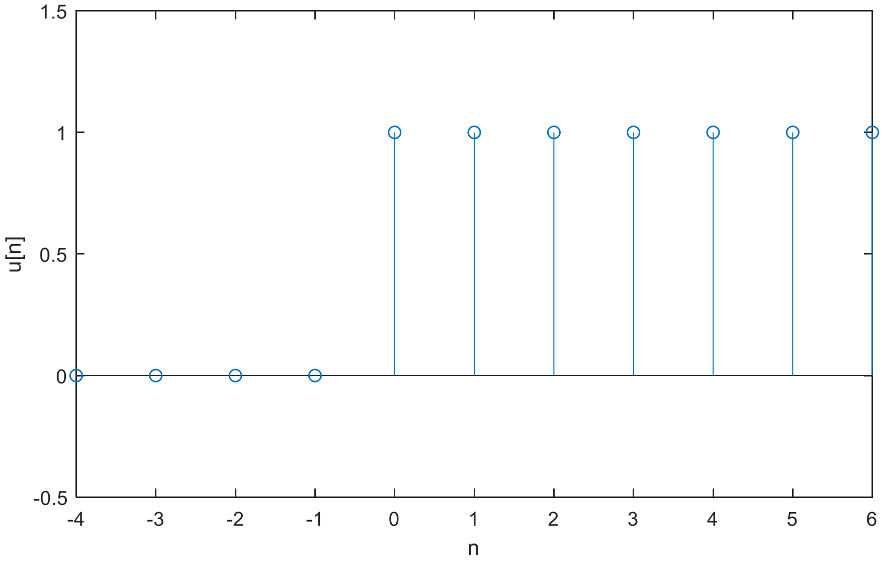
Unit-step signal turns on at n=0 so

$$u[n] = \begin{cases} 1 & for\ n \geq 0 \\ 0 & for\ n < 0 \end{cases}$$

MATLAB code:

```
n = -4:6;
```

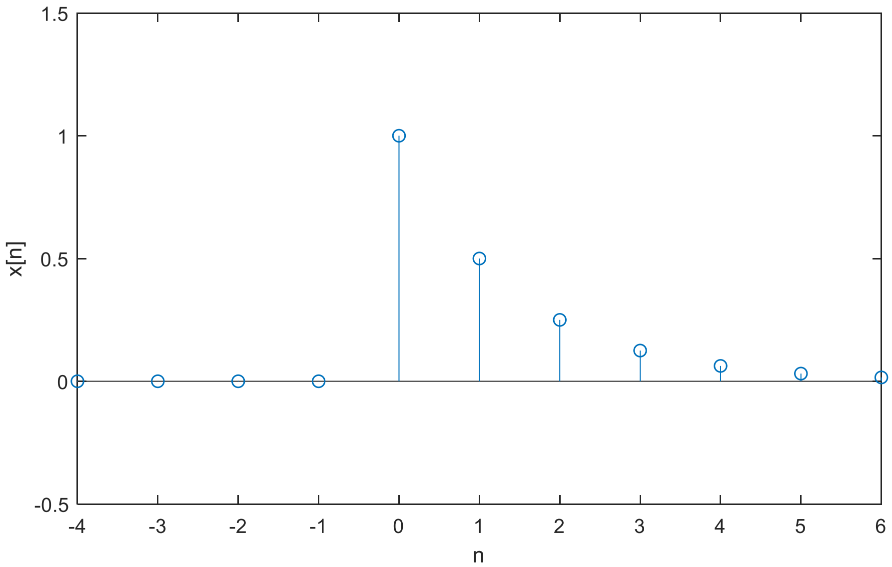
```
u = stepfun(n,0);
stem(n,u)
xlabel('n')
ylabel('u[n]')
ylim([-0.5 1.5])
```



Part b:

MATLAB Code:

```
n = -4:6;
u = stepfun(n,0);
x = (0.5.^n).*u;
stem(n,x)
xlabel('n')
ylabel('x[n]')
ylim([-0.5 1.5])
```



Part c:

In order to calculate y[n], the value of x[n] should be calculated for different parts of discrete-time range.

n	X[n]	X[n-1]	X[n-2]	X[n-3]	Y[n]
-5	0	0	0	0	0
-4	0	0	0	0	0
-3	0	0	0	0	0
-2	0	0	0	0	0
-1	0	0	0	0	0
0	1	0	0	0	1/4
1	1/2	1	0	0	3/8
2	1/4	1/2	1	0	7/16
3	1/8	1/4	1/2	1	15/32
4	1/16	1/8	1/4	1/2	15/64
5	1/32	1/16	1/8	1/4	15/128
6	1/64	1/32	1/16	1/8	15/256
7	1/128	1/64	1/32	1/16	15/512
8	1/256	1/128	1/64	1/32	15/1024
9	1/512	1/256	1/128	1/64	15/2048
10	1/1024	1/512	1/256	1/128	15/4096

Part d:

$$x[n] = a^n u[n]$$

$$x[n-k] = \begin{cases} 0, & n < k \\ a^{n-k} & n \geq k \end{cases}$$

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1}{L}\sum_{k=0}^{n} a^{n-k} = \frac{a^n}{L}\sum_{k=0}^{n} a^{-k} = \frac{a^n}{L}\left(\frac{1-a^{-(n+1)}}{1-a^{-1}}\right) = \frac{1}{L}\left(\frac{a^{n+1}-1}{a-1}\right), & 0 \leq n \leq L-1 \\ \frac{1}{L}\sum_{k=0}^{L-1} a^{n-k} = \frac{a^n}{L}\sum_{k=0}^{L-1} a^{-k} = \frac{a^n}{L}\left(\frac{1-a^{-L}}{1-a^{-1}}\right) = \frac{a^{n-L+1}}{L}\left(\frac{a^L-1}{a-1}\right), & n \geq L \end{cases}$$

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1}{L}\left(\frac{a^{n+1}-1}{a-1}\right), & 0 \leq n \leq L-1 \\ \frac{a^{n-L+1}}{L}\left(\frac{a^L-1}{a-1}\right), & n \geq L \end{cases}$$

Problem 4-

Prologue: For a discrete-time finite impulse response (FIR) filter with M+1 coefficients, the values of the coefficients are equal to the impulse response *h*[*n*]. Given input *x*[*n*], the output *y*[*n*] is given by

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

This formula determines *y*[*n*] by computing the discrete-time convolution of *x*[*n*] and *h*[*n*].

Deconvolution attempts to determine *h*[*n*] when knowing the input *x*[*n*] and the output *y*[*n*].

Application. An application is in determining the acoustic response of a concert hall. One places an audio speaker on stage and a microphone at one of the seats at head height. A laptop controls the discrete-time signal being played over the audio speaker *x*[*n*] and records the output of the microphone in discrete-time as *y*[*n*]. The values computed for *h*[*n*] give a model for the acoustic response of the room. That is, given an audio signal *x*[*n*], we can compute what a person in the concert hall would hear by convolving *h*[*n*] and *x*[*n*]. This emulation of a concert hall is available on certain audio playback systems.

Approach. There are many methods for deconvolution, i.e. determining *h*[*n*] when knowing the input *x*[*n*] and the output *y*[*n*]. The method below uses the convolution formula for an FIR filter to compute the impulse response *h*[*n*]:

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

Assuming that *h*[*n*] and *x*[*n*] are causal signals, i.e. their amplitude values are zero when *n* < 0, the formula for the first output sample *y*[0] gives us one equation in one unknown *h*[0] because we know the values of *x*[0] and *y*[0]:

$$y[0] = h[0]x[0]$$

We then solve for *h*[0], which works as long as *x*[0] is not zero. The next output sample gives us one equation in one unknown *h*[1]:

$$y[1] = h[0]x[1] + h[1]x[0]$$

We then solve for *h*[1], which works as long as *x*[0] is not zero.

Solution:

Part a:

$$x[n] = u[n]$$
$$y[n] = u[n-1]$$

Using the formula on prologue, the value of h[n] can be calculated.

$$y[0] = h[0]x[0]$$

So the value of h[0] can be calculated as: h[0] = 0/1 = 0

$$y[1] = h[0]x[1] + h[1]x[0]$$
$$h[1] = \frac{1-0}{1} = 1$$

$$y[2] = h[0]x[2] + h[1]x[1] + h[2]x[0]$$
$$h[2] = \frac{1-0-1}{1} = 0$$

n	x[n]	y[n]	h[n]
0	1	0	0
1	1	1	1
2	1	1	0

We've stopped calculating values for *h*[*n*] to see if we've finished. We can now compute the convolution of *h*[*n*] and *x*[*n*] = u[n] to see if we get y[n] = u[n-1]

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = x[n-1] = u[n-1]$$

Now, if we place x[n] = δ[n], the output of system is y[n] = h[n]

$$h[n] = \sum_{k=0}^{M} h[k]\delta[n-k] = \delta[n-1]$$

Part b:

$$x[n] = u[n]$$
$$y[n] = \delta[n]$$

$$h[0] = \frac{y[0]}{x[0]} = 1$$
$$h[1] = \frac{y[1] - h[0]x[1]}{x[0]} = -1$$
$$h[2] = \frac{y[2] - h[0]x[2] - h[1]x[1]}{x[0]} = 0$$

n	x[n]	y[n]	h[n]
0	1	1	1
1	1	0	-1
2	1	0	0

We've stopped calculating values for *h*[*n*] to see if we've finished. We can now compute the convolution of *h*[*n*] and *x*[*n*] = u[n] to see if we get y[n] = δ[n]

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = x[n] - x[n-1] = u[n] - u[n-1] = \delta[n]$$

Now, if we place x[n] = δ[n], the output of system is y[n] = h[n]

$$h[n] = \sum_{k=0}^{M} h[k]\delta[n-k] = \delta[n] - \delta[n-1]$$

Part c:

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$
$$y[n] = \delta[n-1]$$

$$h[0] = \frac{y[0]}{x[0]} = 0$$
$$h[1] = \frac{y[1] - h[0]x[1]}{x[0]} = \frac{1-0}{1} = 1$$
$$h[2] = \frac{y[2] - h[0]x[2] - h[1]x[1]}{x[0]} = \frac{0-0-0.5}{1} = -0.5$$
$$h[3] = \frac{y[3] - h[0]x[3] - h[1]x[2] - h[2]x[1]}{x[0]} = \frac{0-0-0.25+0.25}{1} = 0$$

n	x[n]	y[n]	h[n]
0	1	0	0
1	1/2	1	1
2	1/4	0	-0.5
3	1/8	0	0

We've stopped calculating values for *h*[*n*] to see if we've finished. We can now compute the convolution of *h*[*n*] and $x[n] = \left(\frac{1}{2}\right)^n u[n]$ to see if we get y[n] = δ[n-1]

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = x[n-1] - \frac{1}{2}x[n-2] = \left(\frac{1}{2}\right)^{n-1} u[n-1] - (\frac{1}{2})(\frac{1}{2})^{n-2}u[n-2] = \left(\frac{1}{2}\right)^{n-1} \delta[n-1] = \delta[n-1]$$

Now, if we place x[n] = δ[n], the output of system is y[n] = h[n]

$$h[n] = \sum_{k=0}^{M} h[k]\delta[n-k] = \delta[n-1] - \frac{1}{2}\delta[n-2]$$