EE 313 Linear Signals & Systems (Fall 2018)

Solution Set for Homework #6 on Frequency Responses and Z-Transforms

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1. Solution:

a) Find the impulse response of a four-point averaging filter

$$x[n] = \delta[n] \rightarrow h[n] = \frac{1}{4}\left(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]\right)$$

In order to define δ[*n*] in MATLAB, `tripuls(n)` signal function can be used, where n is a discrete-time input, or we can represent *h*[*n*] as a vector of values `[1/4 1/4 1/4 1/4 ]`.

b) The Dirichlet function represents the frequency response of the filter in the following format.

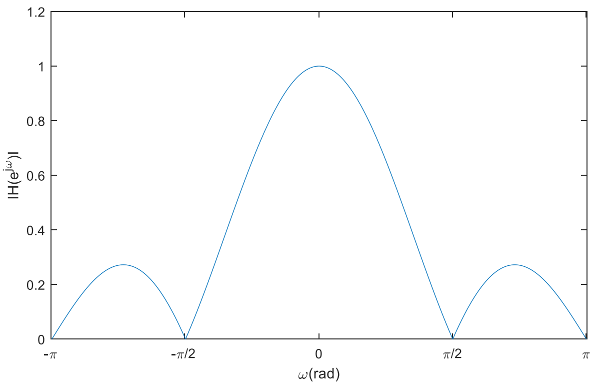
$$H\left(e^{j\hat{\omega}}\right) = D_L\left(e^{j\hat{\omega}}\right)e^{-j\hat{\omega}(L-1)/2} \text{ where } D_L\left(e^{j\hat{\omega}}\right) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

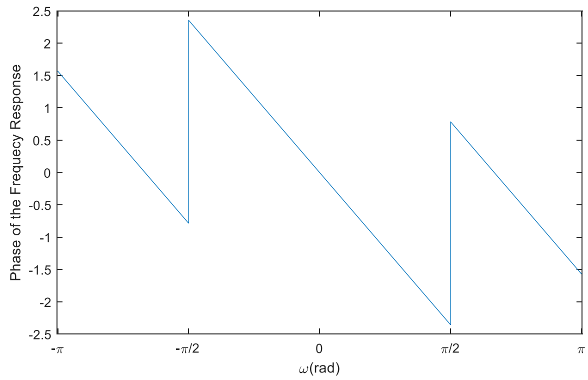
For a four-point averaging filter, *L*=4

$$H\left(e^{j\hat{\omega}}\right) = \frac{\sin(2\hat{\omega})}{4\sin(\hat{\omega}/2)}e^{-j3\hat{\omega}/2}$$

c) In the frequency response that is shown by Dirichlet function, amplitude of filter is equal to $D_L(e^{j\omega})$. It should be considered that $D_L(e^{j\omega})$ can be negative or positive, so the frequency response magnitude is:

$$\left|H\left(e^{j\hat{\omega}}\right)\right| = \left|\frac{\sin(2\hat{\omega})}{4\sin(\hat{\omega}/2)}\right|$$





MATLAB Code:

```
w = -pi:1/10000:pi;
 H = sin(2*w)./(4*sin(w./2)).*exp(-j*3/2*w);
M = abs(H);
 phi = angle (H);
 plot (w,M)
xlim ([-pi pi])
ylim ([0 1.2])
xlabel ('\omega(rad)')
```

```
ylabel ('Magnitude’)
figure
plot (w,phi)
xlim ([-pi pi])
 xlabel ('\omega(rad)')
 ylabel ('Phase')
```

An *L*-point averaging filter has *L*-1 zeros in the magnitude response over one period of the discrete-time frequency domain, i.e. ω in the interval $(-\pi, \pi]$, at discrete-time frequencies $2\pi/L$, $2(2\pi/L)$, etc., and their negative counterparts. For the 4-point averging filter, the zeros are at discrete-time frequencies $-2\pi/4$, $2\pi/4$, and $2(2\pi/4)$, i.e. $-\pi/2$, $\pi/2$, and π.

The phase response is a line of slope -3/2. When the amplitude function goes negative, we multiply it by -1 to make the amplitude value positive and the phase term by -1 = exp(j π) which causes a sudden jump in phase by π or $-\pi$ at the frequency of a zero. A zero in the magnitude response means that the frequency will not passs through the filter.

d) For finding the response to this input, we should calculate frequency response at frequencies that are present in x[n].

$$y[n] = 5H(e^{j0}) + 4\left|H(e^{j0.2\pi})\right|\cos\left(0.2\pi n + \angle H(e^{j0.2\pi})\right) + 3\left|H(e^{j0.5\pi})\right|\cos\left(0.5\pi n + \angle H(e^{j0.5\pi})\right)$$

$$H(e^{j0}) = 1$$

$$H(e^{j0.2\pi}) = 0.7694e^{-j0.3\pi}$$

$$H(e^{j0.5\pi}) = 0$$

$$y[n] = 5 + 3.077\cos\left(0.2\pi n - 0.3\pi\right)$$

e)

$$x_1[n] = x[n]u[n]$$

$$x_1[n] = \begin{cases} x[n] & for \quad n \geq 0 \\ 0 & for \quad n < 0 \end{cases}$$

$$y_1[n] = \frac{1}{4}\left(x_1[n] + x_1[n-1] + x_1[n-2] + x_1[n-3]\right)$$

Hence, $y_1[n] = y[n]$ for $n \geq 3$.

2. At the first stage, the continuous-time input has been received and converted to a discrete-time signal. After passing through the four-point averaging filter, similar to Problem 6.14, it is converted to a continuous-time output.

First we should convert the signal to a discrete time with f_s = 1000 Hz.

$$x(t) = 10 + 8\cos(200\pi t) + 6\cos(500\pi t + \pi/4)$$

$$t = nT_s = \frac{n}{1000} \Rightarrow x[n] = 10 + 8\cos(0.2\pi n) + 6\cos(0.5\pi n + \pi/4)$$

In Problem 6.14, we calculated the frequency response for ω = 0, 0.2π, 0.5π, so we can calculate $y[n]$, which is the output of averaging filter.

$$y[n] = 10H(e^{j0}) + 8\left|H(e^{j0.2\pi})\right|\cos\left(0.2\pi n + \angle H(e^{j0.2\pi})\right)$$
$$+6\left|H(e^{j0.5\pi})\right|\cos\left(0.5\pi n + \pi/4 + \angle H(e^{j0.5\pi})\right)$$

$$y[n] = 10 + 6.156\cos\left(0.2\pi n - 0.3\pi\right)$$

And finally the output is:

$$t = \frac{n}{1000} \Rightarrow y(t) = 10 + 6.156\cos\left(200\pi t - 0.3\pi\right)$$

3. a) $y[n] = x[n] + x[n-1]$ for n > 0 and x[-1] as a necessary condition for the system to be at rest. The impulse response is:

$h[n] = \delta[n] + \delta[n-1]$

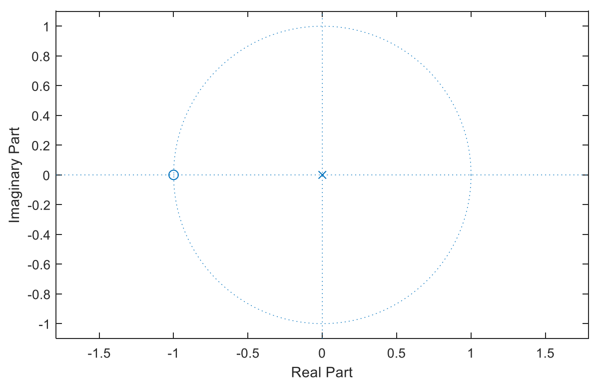
By performing z-transform we can calculate transfer function:

$$H(z) = 1 + z^{-1}$$
$$H(z) = \frac{z+1}{z}$$

The pole (root of the denominator) is at z = 0, and zero (root of the nominator) is at z = -1. Using zplane, we can plot zeros and poles.

```
zplane([1 1])
```

In the following plot, pole is shown by × and zero is depicted by o; hence, the system has one pole in z = 0 and one zero at z = -1.



b)

$$y[n] = x[n] - x[n-1]$$
$$h[n] = \delta[n] - \delta[n-1]$$

Transfer function is:

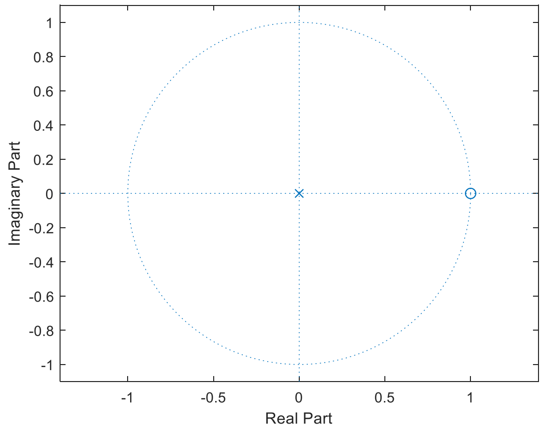
$$H(z) = 1 - z^{-1}$$

$$H(z) = \frac{z-1}{z}$$

Therefore, system has one pole at z = 0, and one zero at z = 1.

MATLAB code:

zplane([1 -1])



c)

$$y[n] = x[n] - 2x[n-1] + x[n-2]$$

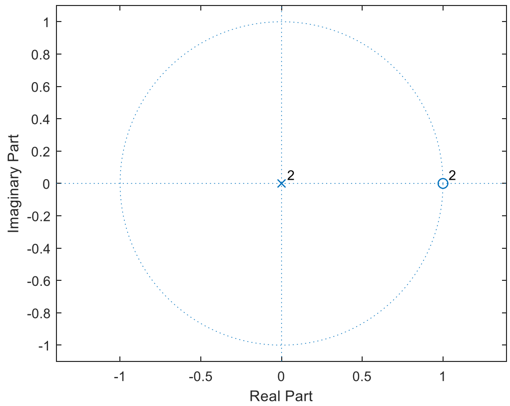
$$h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$$

$$H(z) = 1 - 2z^{-1} + z^{2}$$

$$H(z) = \frac{z^2 - 2z + 1}{z} = \frac{(z-1)^2}{z}$$

System has two poles at z = 0, and two zeros at z = 1:

```
zplane([1 -2 1])
```



4. a) Transforming H(z) to time-domain:

$$h[n] = 1 - 3\delta[n-2] + 2\delta[n-3] + 4\delta[n-6]$$

h[n] is a causal signal with non-zero extent from index 0 to index 6, and x[n] is a causal signal with non-zero extent from index 0 to index 4. Since y[n] is the convolution of h[n] and x[n], y[n] is a causal signal with non-zero extent from n = 0 to n = 10. So, $N_1 = 0$ and $N_2 = 10$.

b) The *z*-transform of the convolution of two signals h[n] and x[n] in the discrete-time domain becomes the product H(z) X(z) in the z-domain. When h[n] and x[n] are of finite length, each becomes a polynomial in z^{-1} in the z-domain, and computing H(z) X(z) becomes a multiplication of two polynomials. The Matlab command conv implements convolution via polynomial multiplication.

$$x[n] = 2\delta[n] + \delta[n-1] - 2\delta[n-2] + 4\delta[n-4]$$

$$X(z) = 2 + z^{-1} - 2z^{-2} + 4z^{-4}$$

$$Y(z) = X(z)H(z) = \left(2 + z^{-1} - 2z^{-2} + 4z^{-4}\right)\left(1 - 3z^{-2} + 2z^{-3} + 4z^{-6}\right)$$

$$Y(z) = 2 + z^{-1} - 8z^{-2} + z^{-3} + 12z^{-4} - 4z^{-5} - 4z^{-6} + 12z^{-7} - 8z^{-8} + 16z^{-10}$$

$$y[n] = 2\delta[n] + \delta[n-1] - 8\delta[n-2] + \delta[n-3] + 12\delta[n-4] - 4\delta[n-5] - 4\delta[n-6] + 12\delta[n-7] - 8\delta[n-8] + 16\delta[n-10]$$

```
h = [2 1 -2 0 4];
x = [1 0 -3 2 0 0 4];
conv(h, x)
% Answer is [2 1 -8 1 12 -4 -4 12 -8 0 16]
```