

Solution Set for Homework #7 on Infinite Impulse Response (IIR) Filters
CORRECTED

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Prolog for the Solution Set. Both discrete-time finite impulse response (FIR) filters and discrete-time infinite impulse response (IIR) filters are widely used in practice. They are both used in speech and audio processing. Speech compression/decompression algorithms on smart phones use IIR filters. Splitting the audible range into sub-woofer (20-200 Hz), woofer (200-2000 Hz) and tweeter (2000-20000 Hz) bands in audio speakers is implemented using IIR filters. FIR filters are more common than IIR filters in image and video processing; however, IIR filters are used in image display and printing. Analog-to-digital converters use IIR filters, and digital-to-analog converters use both FIR and IIR filters. Both FIR and IIR filters are used in communication systems.

As the passband becomes more and more narrow, the number of FIR coefficients increases at a much faster pace vs. the number of IIR coefficients. For example, for a bandpass filter that passes woofer frequencies of 200 Hz to 2000 Hz in a discrete-time audio signal sampled at 48000 Hz would require an FIR filter with 4352 coefficients or an IIR filter with 33 coefficients (obtained from 16 poles, 16 zeros, and 1 gain). Each coefficient represents a multiplication when computing an output sample; hence, the woofer FIR filter requires 100 times more multiplications than the woofer IIR filter.

When we had analyzed the frequency response of the averaging filter on homework #6, tuneup #6, and tuneup #7, we had seen that phase response is linear vs. frequency. Lecture slide 9-9 derives the magnitude and phase response for the two-point averaging filter. A linear phase response corresponds to a fixed delay (called the group delay) through the filter. See lecture slide 9-8. Although we didn't have the chance to go over lecture slides 9-8 and 9-9, we did see a linear phase response for a 3-point lowpass filter on lecture slide 9-6. Also, the bandpass FIR filter based on the Hamming window in mini-project #2 had linear phase (constant group delay).

FIR filters can be designed to have linear phase over all frequencies. Although this is not possible for IIR filters, IIR filters can be designed to have approximate linear phase over the passband. Audio, imaging, and certain communication systems are very sensitive to phase distortion.

IIR filters have one significant drawback— when implemented, they can become bounded-input bounded-output (BIBO) unstable even though their design is BIBO stable. FIR filters are always BIBO stable. Please see lecture slides 11-12 and 11-13 and Handout H on BIBO Stability.

Problem 1

Prolog: This problem asks you to analyze seven linear time-invariant (LTI) filters in the frequency and z domains, and match their magnitude response to one of six example responses. This problem is intended to help you make connections between the time, frequency and z domains. In particular, the problem is intended to build experience on the impact of poles and zeros in the transfer function in the z -domain on the magnitude response of a filter. Each LTI filter is observed for $n \geq 0$. All of the initial conditions are zero as a necessary condition for LTI to hold.

MATLAB code is given at the end of the problem solution.

Concerning the connections between the time, frequency and z domains for discrete-time filters, I recorded a YouTube video in spring 2014 for the EE 445S Real-Time Digital Signal Processing Lab course. Please watch from the 1:29 mark to the 22:25 mark and from 43:01 to the end (50:51) at

<https://www.youtube.com/watch?v=WWEKNvvJBvs&list=PLajppqXMef2ZHIKM4vpwHIAWyRmw3TtSf>

Poles or zeros at the origin in the z-domain have no impact on the magnitude response between the distance from any point on the unit circle $z = e^{j\hat{\omega}}$ to the origin is 1 for all values of the discrete-time frequency $\hat{\omega}$. When a group of poles is separate in angle from a group of zeros, the angles of poles near the unit circle indicate a passband of the filter and the angles of zeros on or near the unit circle indicate a stopband of the filter. This is the case for all of the filters below except filter S2. Filter S2 has one pole and one zero at the same angle. Their radii are inversely related, and this gives an all-pass filter. Please see Handout I on All-Pass Filters at

<http://users.ece.utexas.edu/~bevans/courses/signals/handouts/Appendix%20I%20All%20Pass%20Filters.pdf>

Solutions

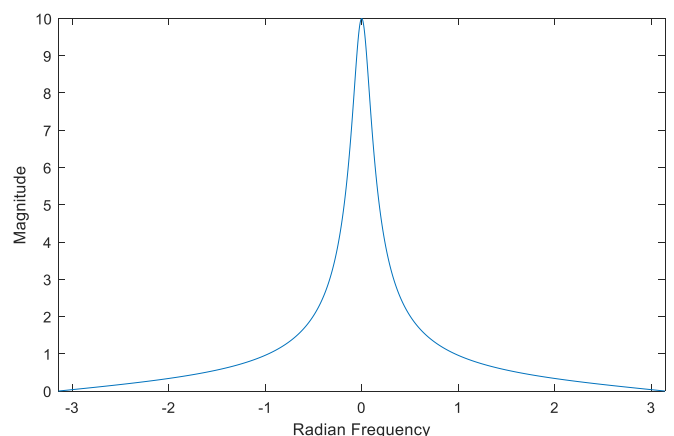
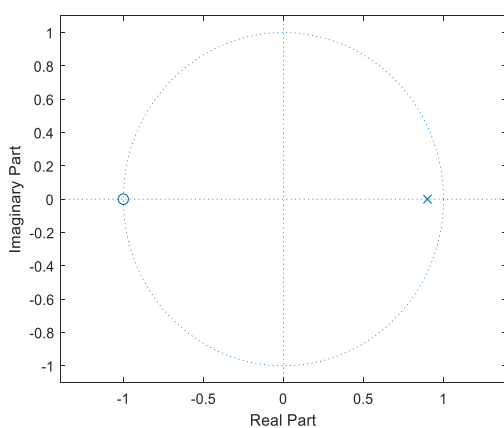
Filter S1 is given as a difference equation. This problem was the subject of Tune-Up Tuesday #9.

$$y[n] = 0.9y[n-1] + \frac{1}{2}x[n] + \frac{1}{2}x[n-1] \Rightarrow y[n] - 0.9y[n-1] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

$$Y(z) - 0.9z^{-1}Y(z) = \frac{1}{2}X(z) + \frac{1}{2}z^{-1}X(z) \rightarrow Y(z)(1 - 0.9z^{-1}) = X(z)\left(\frac{1}{2} + \frac{1}{2}z^{-1}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{1}{2}(1 + z^{-1})}{1 - 0.9z^{-1}} = \frac{\frac{1}{2}(z + 1)}{z - 0.9} = \frac{1}{2} \left(\frac{z + 1}{z - 0.9} \right)$$

To calculate zeros and poles, we calculate roots of numerator and denominator, respectively.



Zeros: $z + 1 = 0 \Rightarrow z = -1$. Zero on the unit circle at angle π indicates that frequencies at $-\pi$ and π rad/sample are zeroed out and are in the stopband.

Poles: $z - 0.9 = 0 \Rightarrow z = 0.9$ A pole near the unit circle indicates that a peak in the magnitude response occurs at the pole angle, which is 0 rad/sample. The passband is centered at 0 rad/sample.

Frequency response matches figure A. Frequency response shows that the frequencies around zero are passed and the magnitude decreases for higher frequencies, hence, S1 is a low-pass filter.

Filter S2 is also given as a difference equation.

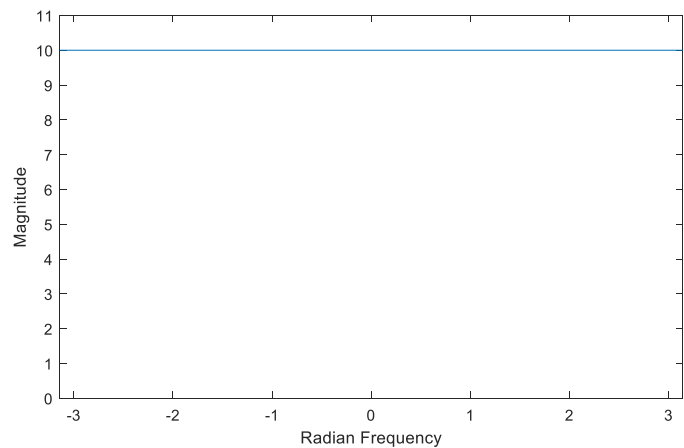
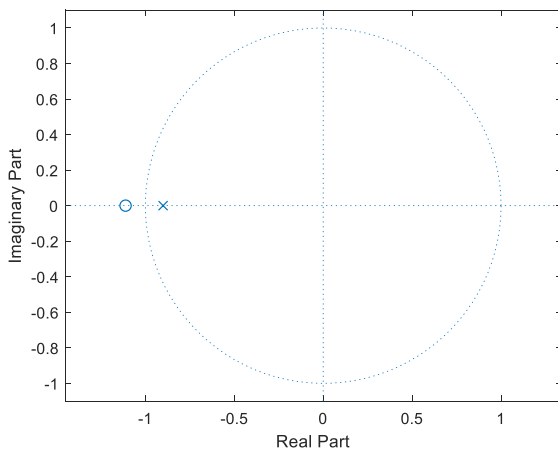
$$y[n] = -0.9y[n-1] + 9x[n] + 10x[n-1] \quad \Rightarrow \quad y[n] + 0.9y[n-1] = 9x[n] + 10x[n-1]$$

$$Y(z) + 0.9z^{-1}Y(z) = 9X(z) + 10z^{-1}X(z) \rightarrow Y(z)(1 + 0.9z^{-1}) = X(z)(9 + 10z^{-1})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{9 + 10z^{-1}}{1 + 0.9z^{-1}} = \frac{9z + 10}{z + 0.9}$$

Zero: $9z + 10 = 0 \Rightarrow z = -10/9 = -1.111$. Radius is the inverse of the radius of the pole.

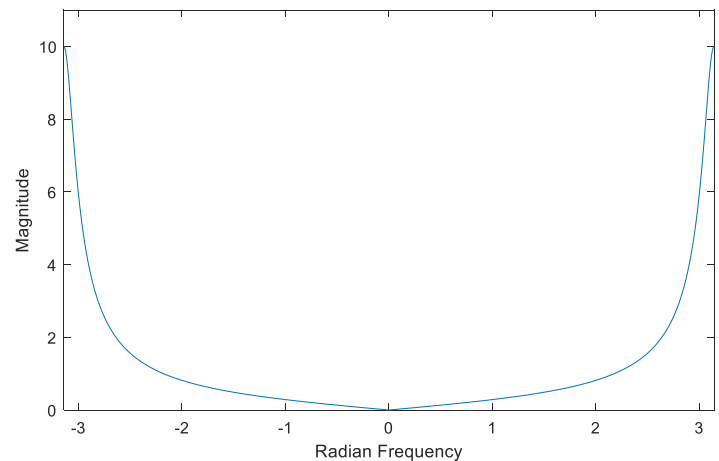
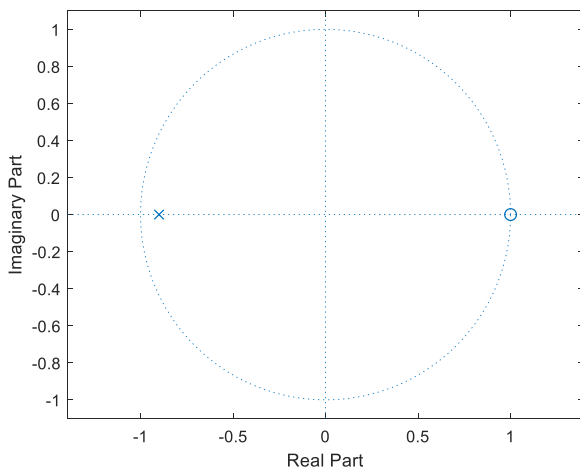
Pole: $z + 0.9 = 0 \Rightarrow z = -0.9$. Pole and zero are at the same angle will interact. See the prolog.



The frequency response matches figure D. This filter passes all frequencies with the same magnitude, i.e. magnitude = 10, so S2 is an all-pass filter.

Filter S3 is given by its transfer function in the z domain:

$$H(z) = \frac{\frac{1}{2}(1 - z^{-1})}{1 + 0.9z^{-1}} = \frac{\frac{1}{2}(z - 1)}{z + 0.9}$$



Zero: $z - 1 = 0 \Rightarrow z = 1$. Zero on the unit circle at angle 0 indicates that frequency at 0 rad/sample is zeroed out and is in the stopband. In fact, the stopband is centered at 0 rad/sample.

Pole: $z + 0.9 = 0 \Rightarrow z = -0.9$. A pole near the unit circle indicates that a peak in the magnitude response occurs at the pole angle, which is π rad/sample. The passband is centered at π rad/sample

This plot matches figure B, which is a highpass filter.

Filter S4 is given as a difference equation:

$$y[n] = \frac{1}{4}x[n] + x[n-1] + \frac{3}{2}x[n-2] + x[n-3] + \frac{1}{4}x[n-4]$$

$$Y(z) = \frac{1}{4}X(z) + z^{-1}X(z) + \frac{3}{2}z^{-2}X(z) + z^{-3}X(z) + \frac{1}{4}z^{-4}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4} + z^{-1} + \frac{3}{2}z^{-2} + z^{-3} + \frac{1}{4}z^{-4} = \frac{\frac{1}{4}z^4 + z^3 + \frac{3}{2}z^2 + z + \frac{1}{4}}{z^4}$$

We can calculate the roots of the numerator by using roots() in MATLAB.

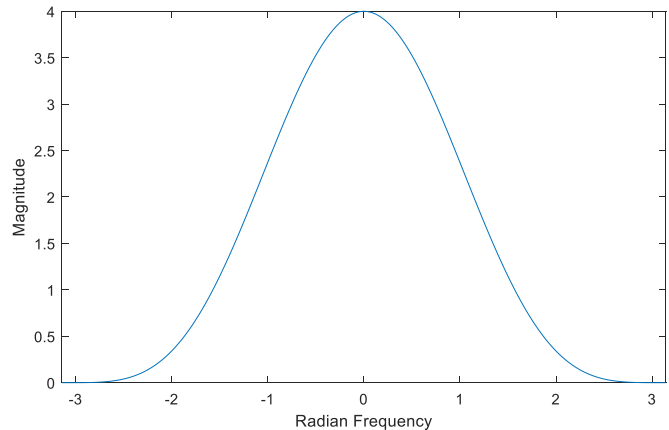
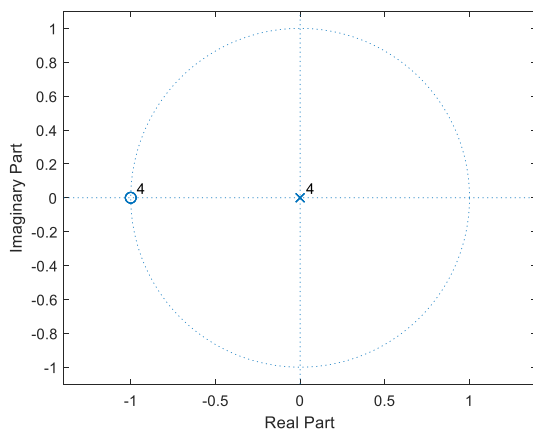
`p = [1/4 1 3/2 1 1/4];`

`roots(p)`

MATLAB find four unique roots close to -1:

$$\text{Zeros: } \frac{1}{4}z^4 + z^3 + \frac{3}{2}z^2 + z + \frac{1}{4} = 0 \Rightarrow z_1 = -1.0002, z_{2,3} = -1 \pm j0.0002, z_4 = -0.9998$$

The correct answer is a repeated root at $z = -1$, which is correctly indicated in the pole-zero plot.



Having four repeated zeros at angle of π means that the reduction in magnitude at frequencies near π rad/sample are raised to the fourth power. The small magnitude values become much smaller.

Poles: $z^4 = 0 \Rightarrow z = 0$ Four poles at $z=0$. Poles at the origin in the z domain have no effect on the magnitude response.

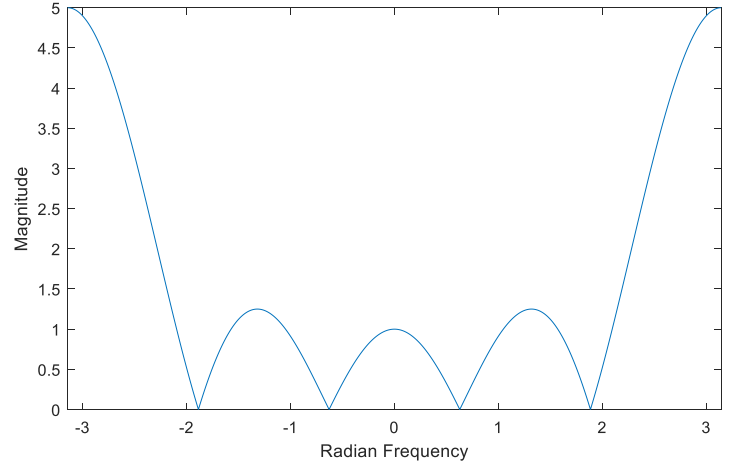
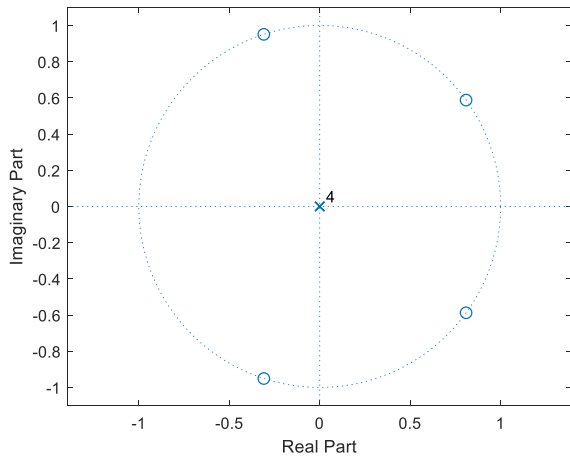
Frequency response is similar to Figure F, which is a lowpass filter.

Filter S5 is given as a transfer function in the z domain:

$$H(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4} = \frac{z^4 - z^3 + z^2 - z^1 + 1}{z^4}$$

Zeros: $z^4 - z^3 + z^2 - z^1 + 1 = 0 \Rightarrow z_{1,2} = -0.3090 \pm j0.9511, \quad z_{3,4} = 0.8090 \pm j0.5878$

Poles: Four poles at $z=0$.



This plot is shown in Figure E. According to the frequency response, we have a highpass filter, because it has high gain at higher frequencies and low gain at lower frequencies. For S3, we had another high pass filter, there the magnitude in stop band was very low while for S5 we can see that even at 0, considerable magnitude can be detected.

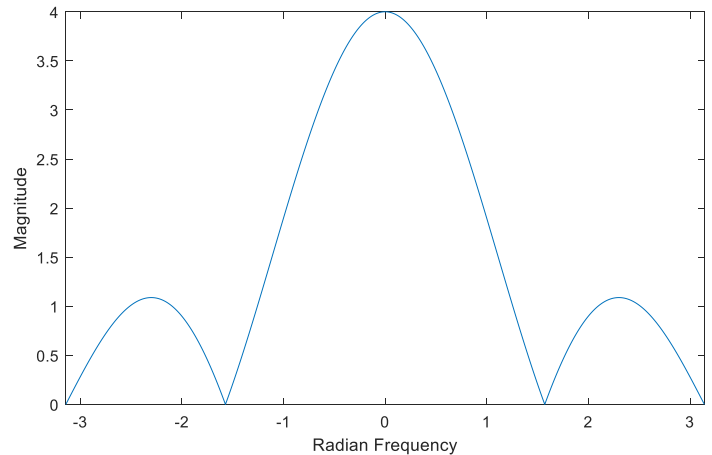
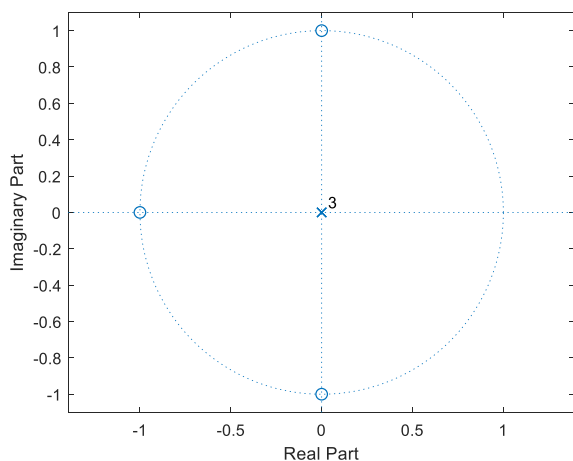
Filter S6 is given as a difference equation:

$$y[n] = \sum_{k=0}^3 x[n-k] \Rightarrow Y(z) = \sum_{k=0}^3 X(z)z^{-k} = X(z) \sum_{k=0}^3 z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^3 z^{-k} = 1 + z^{-1} + z^{-2} + z^{-3} = \frac{z^3 + z^2 + z + 1}{z^3}$$

Zeros: $z^3 + z^2 + z + 1 = 0 \Rightarrow z_1 = -1, z_{2,3} = \pm j$

Poles: $z^3 = 0 \Rightarrow z = 0$ Three poles at $z = 0$.



The frequency response is like figure C, and it represents a lowpass filter.

Filter S7 is given as a difference equation:

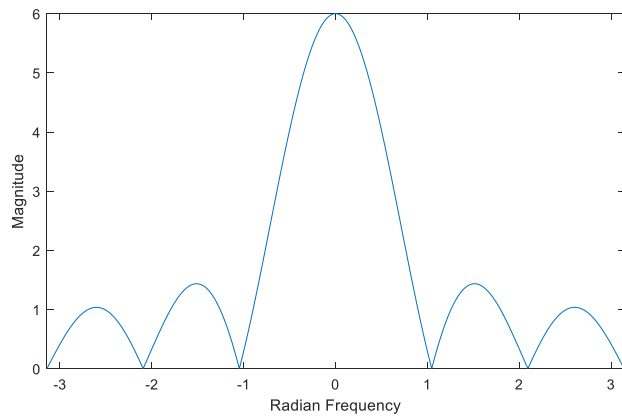
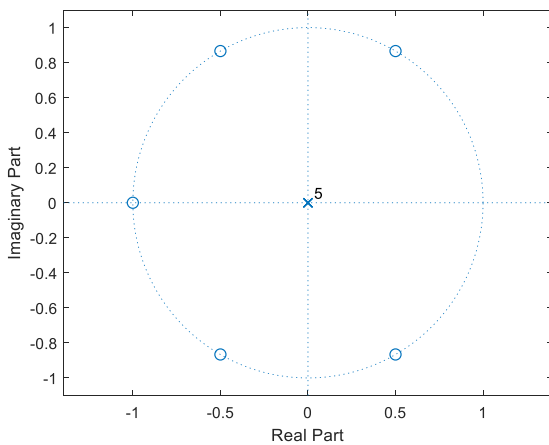
$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] + x[n-5]$$

$$Y(z) = X(z) + z^{-1}X(z) + z^{-2}X(z) + z^{-3}X(z) + z^{-4}X(z) + z^{-5}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} = \frac{z^5 + z^4 + z^3 + z^2 + z + 1}{z^5}$$

Zeros: $z^5 + z^4 + z^3 + z^2 + z + 1 = 0 \Rightarrow z_1 = -1, z_{2,3} = 0.5 \pm j0.866, z_{4,5} = -0.5 \pm j0.866$

Poles: $z^5 = 0 \Rightarrow z = 0$ Five poles at zero



The frequency response shows a lowpass filter, but it does not match any of the six candidate magnitude responses.

MATLAB code:

```
%S1
%feedforwardCoeffs = [ 1/2 1/2 ];
%feedbackCoeffs = [ 1 -0.9 ];
%-----
%S2
%feedforwardCoeffs = [ 9 10 ];
%feedbackCoeffs = [ 1 0.9 ];
%-----
%S3
%feedforwardCoeffs = [ 1/2 -1/2 ];
%feedbackCoeffs = [ 1 0.9 ];
%-----
%S4
%feedforwardCoeffs = [ 1/4 1 3/2 1 1/4 ];
%feedbackCoeffs = 1;
%-----
%S5
%feedforwardCoeffs = [ 1 -1 1 -1 1 ];
%feedbackCoeffs = 1;
%-----
```

```

%S6
%feedforwardCoeffs = [ 1 1 1 1 ];
%feedbackCoeffs = 1;
%-----
%S7
feedforwardCoeffs = [ 1 1 1 1 1 1 ];
feedbackCoeffs = 1;
%-----
figure;
zplane(feedforwardCoeffs, feedbackCoeffs);
W = -pi : 0.001 : pi;
[H, W] = freqz( feedforwardCoeffs, feedbackCoeffs, W );
figure;
plot(W, abs(H));
xlim([-pi pi])
xlabel('Radian Frequency')
ylabel('Magnitude')
    
```

Problem 2: a) This system is formed by cascading two LTI systems. In the z-domain:

$$\begin{cases} W(z) = H_1(z)X(z) \\ Y(z) = H_2(z)W(z) \end{cases} \Rightarrow Y(z) = H_2(z)H_1(z)X(z) = H(z)X(z)$$

$$H(z) = H_2(z)H_1(z)$$

b) $y_1[n] = x[n] + \frac{5}{6}x[n-1]$

$$Y_1(z) = X(z) + \frac{5}{6}z^{-1}X(z)$$

$$H_1(z) = \frac{Y_1(z)}{X(z)} = 1 + \frac{5}{6}z^{-1}$$

$$H(z) = H_2(z)H_1(z) = (1 - 2z^{-1} + z^{-2})\left(1 + \frac{5}{6}z^{-1}\right) = 1 - \frac{7}{6}z^{-1} - \frac{2}{3}z^{-2} + \frac{5}{6}z^{-3}$$

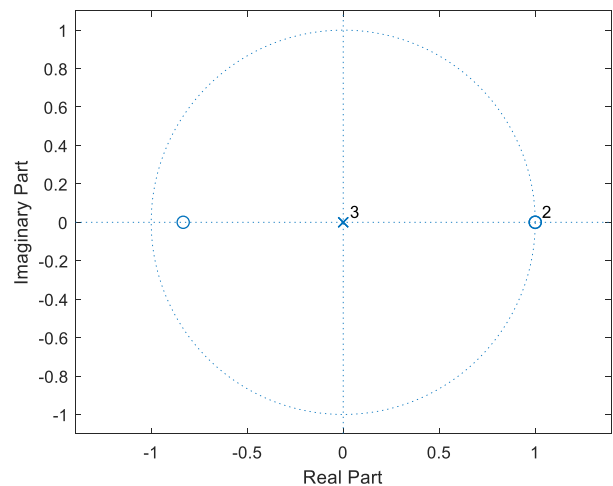
c) $Y(z) = H(z)X(z)$

$$y[n] = x[n] - \frac{7}{6}x[n-1] - \frac{2}{3}x[n-2] + \frac{5}{6}x[n-3]$$

d) MATLAB code

```

h1 = [ 1 5/6 ];
h2 = [ 1 -2 1 ];
h = conv(h2, h1);
zplane(h)
    
```



e) $y[n] = x[n]$ so $Y(z) = X(z) \Rightarrow H(z) = 1$

f) According to part e, $H(z) = 1$ to guarantee input and output are equal. Also, from part (b)

$$H(z) = H_2(z)H_1(z)$$

$$1 = H_2(z) \left(1 + \frac{5}{6}z^{-1} \right)$$

$$H_2(z) = \frac{1}{1 + \frac{5}{6}z^{-1}}$$

g) H_2 will be stable if its poles are inside the unit circle. $H_2(z)$ is inverse of $H_1(z)$, so zeros of $H_1(z)$ are poles of $H_2(z)$. Therefore, zeros of $H_1(z)$ should be inside the unit circle.

Epilog: Parts (f) and (g) concern the design of filter H_2 compensate for the frequency distortion introduced by filter H_1 . Equalizing frequency distortion in an unknown system has many applications, including display/printing of images, calibrating biomedical instrumentation, compensating phase distortion in an analog-to-digital converter, and communication receivers. In a communication receiver, the channel equalizer compensates for frequency distortion in the communication channel as well as in the analog/RF circuits in the transmitter and receiver.