EEE 313 Linear Signals & Systems (Fall 2018)

Solution Set for Homework #8 on Continuous-Time Signals & Systems

By: Mr. Houshang Salimian & Prof. Brian L. Evans

Here are several useful properties of the Dirac delta functional (generalized function):

a) *Unit area*: $\int_{-\infty}^{\infty} \delta(x)dx = 1$

b) *Sifting property:* $\int_a^b f(x)\delta(x - x_0)dx = \begin{cases} f(x_0), x_0 \in [a,b] \\ 0 \quad\quad , otherwise \end{cases}$

c) *Even symmetry:* $\delta(x) = \delta(-x)$

d) *Relationship to the unit step function.* $\frac{d}{dx}u(x) = \delta(x)$.

Here are several comments about bounded-input bounded-output (BIBO) stability:

e) *BIBO Stability: If input x(t) is bounded in amplitude, i.e.* $|x(t)| \leq B$ *for a finite value B, then output y(t) is always bounded in amplitude, i.e.*$|y(t)| \leq B_1$ *for a finite value* B_1*. This definition does not require the system to be LTI.*

f) *BIBO stability for LTI systems: For a continuous-time LTI system with an impulse response h(t), BIBO stability reduces to* $\int_{-\infty}^{\infty}|h(t)|dt < \infty$*. A derivation is given in problem 3 below.*

g) *BIBO stability for FIR filters: From f), it immediately follows that FIR filters are always BIBO stable (if |h(t)| < ∞ for all t). This is also reflected in the fact that all the poles of an FIR filter are at z=0 (inside the unit circle), which implies stability.*

Please see Handout I on *Bounded-Input Bounded-Output Stability* at
http://users.ece.utexas.edu/~bevans/courses/signals/handouts/Appendix%20H%20BIBO%20Stability.pdf

*Convolution: Let c(t) = x(t)*y(t) =>* $c(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)y(t)d\tau$

Problem 1:

a) In this question we can use the following property:

$$\delta(t-t_0) * f(t) = f(t-t_0)$$

$$\delta(t-10) * [\delta(t+10) + 2e^{-t}u(t) + \cos(100\pi t)] = \delta\left((t-10)+10\right) + 2e^{-(t-10)}u(t-10) + \cos(100\pi(t-10))$$

$$= \delta(t) + 2e^{-(t-10)}u(t-10) + \cos(100\pi t - 1000)$$

b) The Dirac delta functional is defined in terms of integration: (a) it has unit area at the origin and (b) has a sifting property. The Dirac delta functional is waiting around to be integrated. Please avoid simplifying expressions involving the Dirac delta that are not being integrated.

$$\int_{-\infty}^{\infty} \cos(100\pi t)\left[\delta(t) + \delta(t-0.002)\right]dt = \cos(0)\int_{-\infty}^{\infty} \delta(t)dt + \cos\left(0.2\pi\right)\int_{-\infty}^{\infty} \delta(t-0.002)dt$$

$$= 1 + \cos(0.2\pi) = 1.809$$

c) The Dirac delta functional is defined in terms of integration: (a) it has unit area at the origin and (b) has a sifting property. The Dirac delta functional is waiting around to be integrated. Please avoid simplifying expressions involving the Dirac delta that are not being integrated

$$\frac{d}{dt}\left[e^{-2(t-2)}u(t-2)\right]=e^{-2(t-2)}\frac{d}{dt}\left(u(t-2)\right)+u(t-2)\frac{d}{dt}\left(e^{-2(t-2)}\right)=e^{-2(t-2)}\delta(t-2)-2e^{-2(t-2)}u(t-2)$$

d)

$$\int_{-\infty}^{t}\cos\left(100\pi\tau\right)\left[\delta\left(\tau\right)+\delta\left(\tau-0.002\right)\right]d\tau=\int_{-\infty}^{t}\left[\cos\left(0\right)\delta(\tau)+\cos(0.2\pi)\delta(\tau-0.002)\right]d\tau$$

$$=\cos\left(0\right)\int_{-\infty}^{t}\delta(\tau)d\tau+\cos\left(0.2\pi\right)\int_{-\infty}^{t}\delta(\tau-0.002)d\tau=u(t)+\cos(0.2\pi)u(t-0.002)=u(t)+0.809u(t-0.002)$$

Problem 2: This is averaging filter (unnormalized). Its output is the average of the previous two seconds of input, the current input value, and the future two seconds of input. If a gain of ¼ had been applied, then we'd have a normalized averaging filter (normalized so that the area of the absolute value of the impulse response is one).

$$y(t)=\int_{t-2}^{t+2}x(\tau)d\tau$$

a)

$$h(t)=\int_{t-2}^{t+2}\delta(\tau)d\tau=\int_{-\infty}^{t+2}\delta(\tau)d\tau-\int_{-\infty}^{t-2}\delta(\tau)d\tau=\int_{-\infty}^{t}\delta(\tau'+2)d\tau'-\int_{-\infty}^{t}\delta(\tau''-2)d\tau''=u(t+2)-u(t-2)$$

Alternate Solution:

$$h(t)=\int_{t-2}^{t+2}\delta(\tau)d\tau=\begin{cases}0, & t<-2\\ 1, & -2\le t<2\\ 0, & t\ge 2\end{cases}$$

$$h(t)=u(t+2)-u(t-2)$$

b) If $|x(t)|\le B\ \ for\ all\ t$, then $|y(t)|=\left|\int_{t-2}^{t+2}x(\tau)d\tau\right|\le\int_{t-2}^{t+2}|x(\tau)|d\tau\le\int_{t-2}^{t+2}Bd\tau=4B$

So, a bounded input generates a bounded output and hence the system is bounded-input bounded-output (BIBO) stable.

A continuous-time LTI system is stable if and only if: $\int_{-\infty}^{\infty}|h(\tau)|d\tau<\infty$

Here, $\int_{-\infty}^{\infty}|h(\tau)|d\tau=\int_{-2}^{2}1d\tau=4$ and the system is BIBO stable.

c) This system is not causal, because current output is dependant to future value of input. For instance at t=1: $y(1)=\int_{-1}^{3}x(\tau)d\tau$ which shows that output at t=1 is related to input values in future, i.e. t = 1 to 3.

Note: A continuous-time, LTI system is causal if and only if, $h(\tau)=0, for\ \ \tau<0.$ In this question, $h(t)=1, for-2<t<0$, which means this system is not causal.

d)

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty}u(\tau+1)\left[u(t-\tau+2)-u(t-\tau-2)\right]d\tau$$

In order to calculate the convolution we should break time domain into three regions.

1st case (No overlap): for $t+2<-1\Rightarrow t<-3$

In this case, $x(\tau) and\ h(t-\tau)$ do not face any overlap, so $y(t)=0$

$$\int_{-\infty}^{\infty}u(\tau+1)\left[u(t-\tau+2)-u(t-\tau-2)\right]d\tau=0$$

2nd case (partial overlap): for $\begin{cases}t+2\geq-1\Rightarrow t\geq-3\\ t-2<-1\Rightarrow t<1\end{cases}\Rightarrow -3\leq t<1$ there is partial overlap between $x(\tau) and\ h(t-\tau)$

$$y(t)=\int_{-1}^{t+2}u(\tau+1)\left[u(t-\tau+2)-u(t-\tau-2)\right]d\tau=\int_{-1}^{t+2}1d\tau=\tau\Big|_{-1}^{t+2}=(t+2)-(-1)=t+3$$

3rd case (complete overlap):

for $t-2\geq-1\Rightarrow t\geq1$

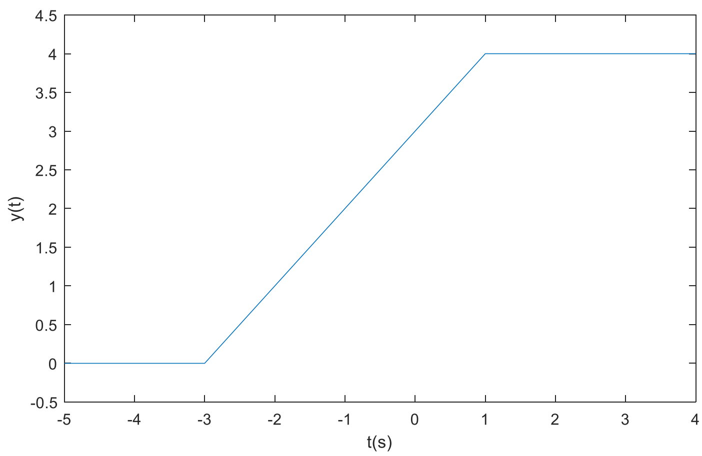
$$y(t)=\int_{t-2}^{t+2}1d\tau=\tau\Big|_{t-2}^{t+2}=(t+2)-(t-2)=4$$

Therefore:

$$y(t)=\begin{cases}0, & t<-3\\ t+3, & -3\leq t<1\\ 4, & t\geq1\end{cases}$$

MATLAB code for plotting output:

```
fs = 8000;
t = -5: 1/fs :4;
yy = zeros(size(t));
yy (t>=-3 & t<1) = t(t>=-3 & t<1)+3; %second case -3 =< t < 1 and y(t) =
t+3
yy(t >= 1) = 4; % third case t >= 1 and y(t) = 4
plot(t,yy)
ylim ([-0.5 4.5])
xlabel ('t(s)')
ylabel ('y(t)')
```



Problem 3:

$$x(t) = u(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} u(t-\tau)\left[\delta(\tau) - 3e^{-3\tau}u(\tau)\right] d\tau = \int_{-\infty}^{\infty} u(t-\tau)\delta(\tau)d\tau - \int_{-\infty}^{\infty} 3e^{-3\tau}u(\tau)u(t-\tau)d\tau = y_1(t) + y_2(t)$$

$$y_1(t) = \int_{-\infty}^{\infty} u(t-\tau)\delta(\tau)d\tau = \int_{-\infty}^{\infty} u(t-0)\delta(\tau)d\tau = u(t)\int_{-\infty}^{\infty} \delta(\tau)d\tau = u(t)$$

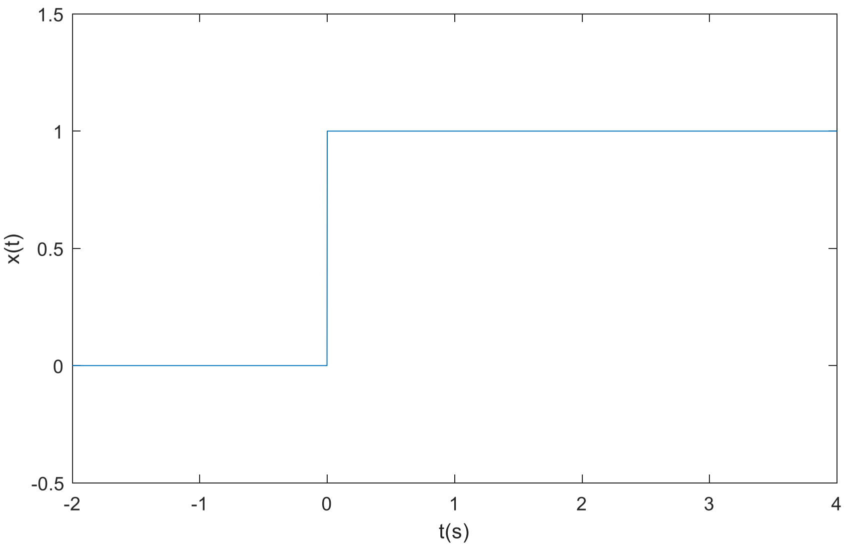
$$y_2(t) = -\int_{-\infty}^{\infty} 3e^{-3\tau}u(\tau)u(t-\tau)d\tau = -\int_{0}^{\infty} 3e^{-3\tau}u(t-\tau)d\tau = \begin{cases} 0, & t < 0 \\ -\int_0^t 3e^{-3\tau}d\tau = e^{-3\tau}\Big|_0^t = e^{-3t} - 1, & t \geq 0 \end{cases}$$

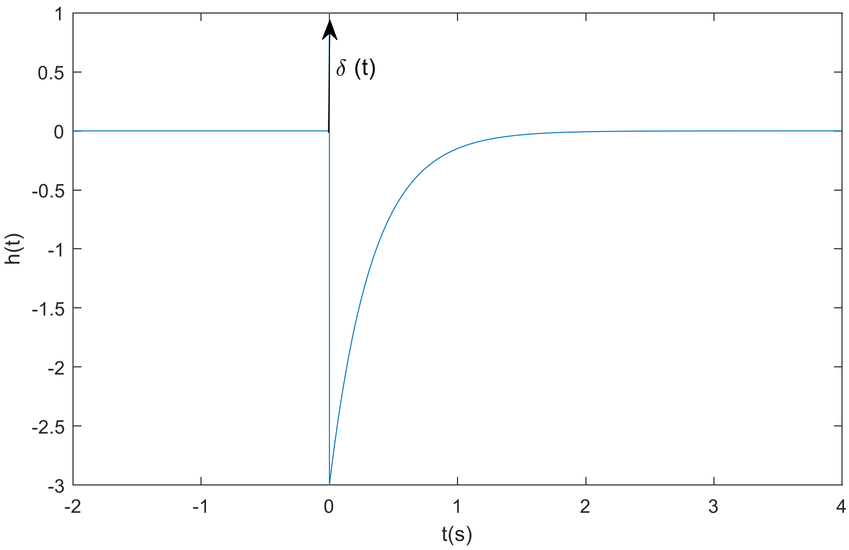
$$y(t) = x(t) * h(t) = y_1(t) + y_2(t) = u(t) + \left[e^{-3t} - 1\right]u(t) = e^{-3t}u(t)$$

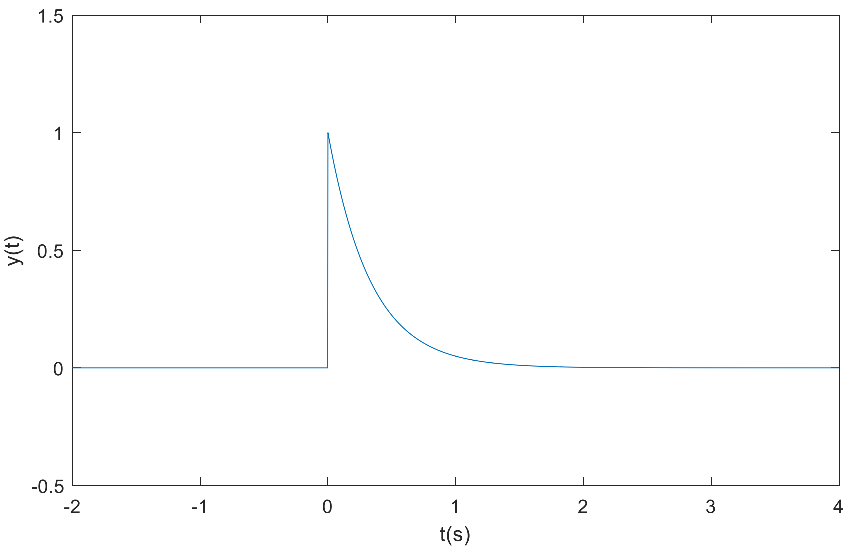
See graphical flip-and-slide convolution on page 6.

MATLAB code:

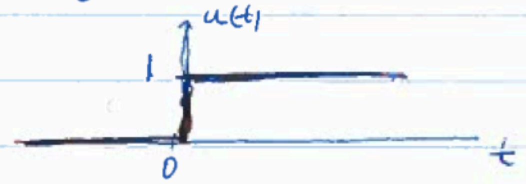
```
clear all
fs = 8000;
t = -2: 1/fs :4;
unitstep = zeros(size(t));
unitstep (t>= 0) = 1; % define unit step function
x = unitstep; % define input x(t) = u(t)
impulse = dirac(t); % define dirac delta function
idx = impulse == Inf;
impulse (idx) = 4;
h = impulse - 3*exp(-3*t).*unitstep; % h(t) is system response
y= exp(-3*t).*unitstep; % y(t) = system's output  for x(t) = u(t)

figure
plot(t,x)
ylim([-0.5 1.5])
xlabel ('t(s)')
ylabel ('x(t)')

figure
plot(t,h)
xlabel ('t(s)')
ylabel ('h(t)')

figure
plot(t,y)
ylim([-0.5 1.5])
xlabel ('t(s)')
ylabel ('y(t)')
```

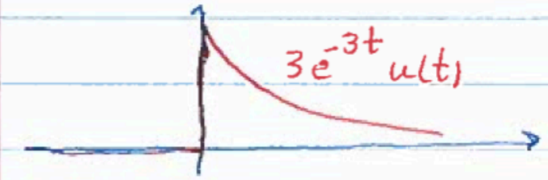

1.5
1
0.5
0
-0.5
x(t)
-2
-1
0
1
2
3
4
t(s)


1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
h(t)
δ (t)
-2
-1
0
1
2
3
4
t(s)

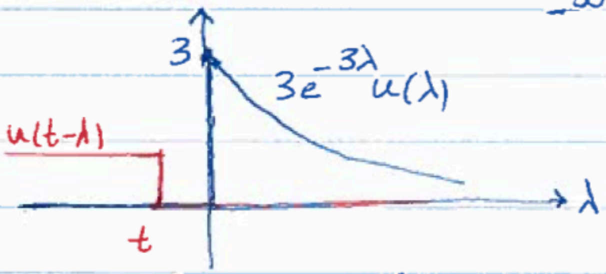

1.5
1
0.5
0
-0.5
y(t)
-2
-1
0
1
2
3
4
t(s)

HW 8.3

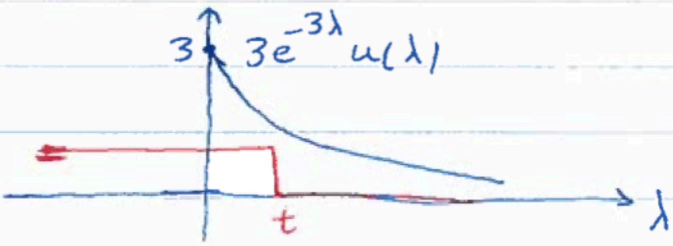




$$(3e^{-3t}u(t)) * u(t) = \int_{-\infty}^{\infty} (3e^{-3\lambda}u(\lambda))\, u(t-\lambda)\, d\lambda$$



Case I: No overlap



Case II: Partial / Complete Overlap

$$3e^{-3t}u(t) * u(t) = \begin{cases} 0 & \text{for } t \leqslant 0 \quad \text{case I} \\ \int_0^t 3e^{-3\lambda}\, d\lambda & \text{for } t > 0 \quad \text{case II} \end{cases}$$

Problem 4:

The impulse response for the first system will be calculated by placing:

$$x(t) = \delta(t) \Rightarrow y(t) = h_1(t)$$

$$h_1(t) = \delta(t) - \delta(t-2)$$

Where the output of first LTI system, w(t), is $w(t) = h_1(t) * x(t)$, and the output of second LTI system is $y(t) = h_2(t) * w(t)$. Here, two systems are connected in cascade:

$$y(t) = h_2(t) * w(t) = h_2(t) * h_1(t) * x(t) = h(t) * x(t)$$

The impulse response for the cascaded systems is:

$$h(t) = h_2(t) * h_1(t) = u(t) * \left[\delta(t) - \delta(t-2)\right] = u(t) * \delta(t) - u(t) * \delta(t-2) = u(t) - u(t-2)$$

MATLAB code:

```
clear all
fs = 8000;
t = -2: 1/fs :4;
unitstep0 = t>= 0;
unitstep2 = t>= 2;
h = unitstep0 - unitstep2; % h(t) is system response
figure
plot(t,h)
ylim ([-0.5 1.5])
xlabel ('t(s)')
ylabel ('h(t)')
```