EE 313 Linear Signals & Systems (Fall 2018)

Solution Set for Homework #9 on Continuous-Time Frequency Response and Fourier Transforms

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Problem 1:

A linear time-invariant (LTI) system has an unknown frequency response $H(j\,\omega)$.

A periodic input signal $x(t)$ with fundamental frequency ω_0 is input to the LTI system.

The Fourier series coefficients for $x(t)$ are given in the problem as

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{1}{\pi}, & k = \pm 1 \\ 0, & k = \pm 2 \\ -\frac{1}{3\pi}, & k = \pm 3 \\ 0, & k = \pm 4 \\ \frac{1}{5\pi}, & k = \pm 5 \end{cases}$$

$$x(t) = \sum_{k=-5}^{5} a_k e^{jk\omega_0 t} = \frac{1}{2} + \frac{2}{\pi}\cos(\omega_0 t) - \frac{2}{3\pi}\cos(3\omega_0 t) + \frac{2}{5\pi}\cos(5\omega_0 t)$$

Due to the system having linear and time-invariant properties, all of the frequency components in the output signal had to be present in the input signal. That is, a linear time-invariant system cannot create new frequencies.

Using LTI system properties, the output signal is simply the sum of the system's response to each frequency component of the input signal:

$$y(t) = \frac{1}{2}H(j0) + \frac{2}{\pi}\cos(\omega_0 t)H(j\omega_0) - \frac{2}{3\pi}\cos(3\omega_0 t)H(j3\omega_0) + \frac{2}{5\pi}\cos(5\omega_0 t)H(j5\omega_0)$$

We can write the frequency response into polar form as $H(j\omega) = |H(j\omega)|e^{j\,\angle H(j\omega)}$:

$$\begin{aligned} y(t) = {} & \frac{1}{2}H(j0) + \frac{2}{\pi}|H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) - \\ & \frac{2}{3\pi}|H(j3\omega_0)|\cos(3\omega_0 t + \angle H(j3\omega_0)) + \\ & \frac{2}{5\pi}|H(j5\omega_0)|\cos(5\omega_0 t + \angle H(j5\omega_0)) \end{aligned}$$

Please see lecture slide 14-6 and *Signal Processing First* Section 10-2.

a) $y(t) = \frac{1}{2}$

The output of system can be obtained by the following formula

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0) a_k e^{jk\omega_0 t}$$

Hence, we can find the value of H(jω) for frequencies that are present in the input.

$$H(jk\omega_0) = \begin{cases} 1, & k = 0 \\ 0, & k = \pm 1, \pm 3, \pm 5 \end{cases}$$

Filter 5 has similar response, so the input has passed through this lowpass filter.

$$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases}$$

b)

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos\left[\omega_0\left(t - \frac{1}{2}\right)\right]$$

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0) a_k e^{jk\omega_0 t} = \frac{1}{2} + \frac{2}{\pi}\cos\left[\omega_0\left(t - \frac{1}{2}\right)\right]$$

$$H\left(jk\omega_0\right) = \begin{cases} 1, & k = 0 \\ e^{-j\omega_0/2}, & k = \pm 1 \\ 0, & k = \pm 3, \pm 5 \end{cases}$$

Filter 6 has this property and will give similar output. This lowpass filter removes frequencies above $3\omega_0/2$ and delays the input by ½ sample. We can obtain the delay by computing the group delay for the filter as follows: $\text{GroupDelay}(\omega) = -\frac{d}{d\omega}\angle H(j\omega) = -\frac{d}{d\omega}\left(-\frac{\omega}{2}\right) = \frac{1}{2}$.

$$H(j\omega) = \begin{cases} e^{-j\omega/2}, & |\omega| < \frac{3}{2}\omega_0 \\ 0, & |\omega| > \frac{3}{2}\omega_0 \end{cases}$$

c)

$$y(t) = \frac{2}{\pi}\cos\left(\omega_0 t\right)$$

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0) a_k e^{jk\omega_0 t} = \frac{2}{\pi}\cos\left(\omega_0 t\right)$$

$$H\left(jk\omega_0\right) = \begin{cases} 1, & k = \pm 1 \\ 0, & k = 0, \pm 3, \pm 5 \end{cases}$$

The original signal has passed through a bandpass filter which removes all frequencies present in the input signal except ω_0. Filter 7 shows a bandpass filter with this property.

$$H(j\omega) = \begin{cases} 1, & \frac{1}{2}\omega_0 < |\omega| < \frac{3}{2}\omega_0 \\ 0, & |\omega| < \frac{1}{2}\omega_0 \; or \; |\omega| > \frac{3}{2}\omega_0 \end{cases}$$

d)

$$y(t) = x(t) - \frac{1}{2} = \frac{2}{\pi}\cos(\omega_0 t) - \frac{2}{3\pi}\cos(3\omega_0 t) + \frac{2}{5\pi}\cos(5\omega_0 t)$$

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0)a_k e^{jk\omega_0 t} = \frac{2}{\pi}\cos(\omega_0 t) - \frac{2}{3\pi}\cos(3\omega_0 t) + \frac{2}{5\pi}\cos(5\omega_0 t)$$

$$H(jk\omega_0) = \begin{cases} 0, & k = 0 \\ 1, & k = \pm 1, \pm 3, \pm 5 \end{cases}$$

This filter passes all frequencies except ω=0, therefore it acts as a highpass filter or a bandpass filter. Filter 1 is a highpass filter that can produce this output.

$$H(j\omega) = \begin{cases} 0, & |\omega| < \frac{1}{2}\omega_0 \\ 1, & |\omega| > \frac{1}{2}\omega_0 \end{cases}$$

e)

$$y(t) = x(t - \frac{1}{2})$$

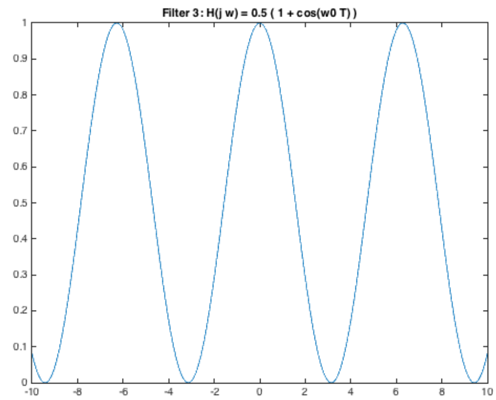
$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0)a_k e^{jk\omega_0 t}$$

$$H(jk\omega_0) = e^{-jk\omega_0/2}$$

Filter 2 gives this response, which is all-pass.

$$H(j\omega) = e^{-j\omega/2}$$

Filters 3 and 4 cannot produce any of output signals.



Filter 3:

$$H(j\omega) = \frac{1}{2}\left[1 + \cos(\omega T_0)\right]$$

$$for\ \omega = k\omega_0 \rightarrow H(jk\omega_0) = \frac{1}{2}\left[1 + \cos(k\omega_0 T_0)\right] = \frac{1}{2}\left[1 + \cos(2\pi k)\right] = 1$$

For $\omega = k\omega_0$, this filter passes all the harmonic frequencies; however, it rejects sub-harmonic frequencies at $\omega = k\omega_0 / 2$. This filter has a periodic magnitude response as shown above.

Filter 4: This filter removes frequencies above $3\omega_0/2$ and is a lowpass filter.

Problem 2:

Properties in Tables 11-2 and 11-3, on pages 338 and 339, of *Signal Processing First* are used for obtaining the Continuous-Time Fourier Transform or Inverse Continuous-Time Fourier Transform.

a)

$$x(t) = \begin{cases} 1, & 0 \le t < 4 \\ 0, & otherwise \end{cases}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{4} 1e^{-j\omega t}dt = -\frac{1}{j\omega}e^{-j\omega t}\Big|_{t=0}^{4} = -\frac{1}{j\omega}\left(e^{-j4\omega} - 1\right) = \frac{e^{-j2\omega}\left(e^{j2\omega} - e^{-j2\omega}\right)}{j\omega} = 2e^{-j2\omega}\frac{\sin(2\omega)}{\omega}$$

Alternate Solution:

$$x(t) = \begin{cases} 1, & 0 \le t < 4 \\ 0, & otherwise \end{cases} = u(t) - u(t-4) = \delta(t-2) * \left(u(t+2) - u(t-2)\right) = \delta(t-2) * rect\left(\frac{t}{4}\right)$$

The following properties are used in this solution.

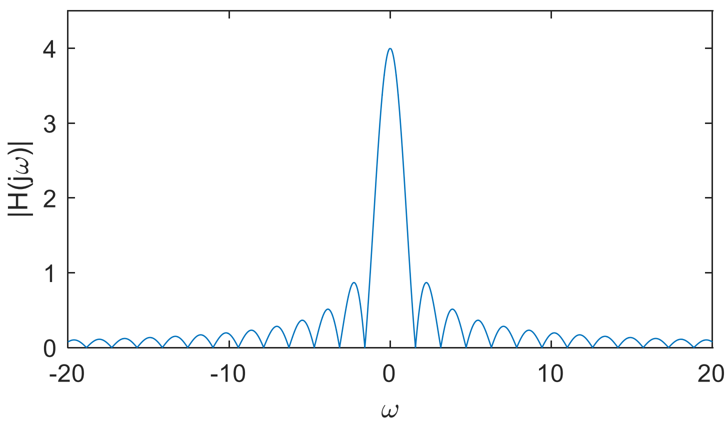
$$x(t) * h(t) \xrightarrow{F} X(j\omega)H(j\omega)$$

$$\delta(t - t_d) \xrightarrow{F} e^{-j\omega t_d}$$

$$u(t + \frac{T}{2}) - u(t - \frac{T}{2}) \xrightarrow{F} 2\frac{\sin(\omega T/2)}{\omega}$$

$$X(j\omega) = 2e^{-j2\omega}\frac{\sin(2\omega)}{\omega}$$

The magnitude response of this signal is shown below. It is lowpass.




```
w = -20:1/10000:20;
X = 2*exp(-j*2*w).*sin(2*w)./w;
plot(w,abs(X))
ylim([0 4.5])
```

b)

$$S(j\omega) = 4\pi\delta(\omega) + 2\pi\delta(\omega - 10\pi) + 2\pi\delta(\omega + 10\pi)$$

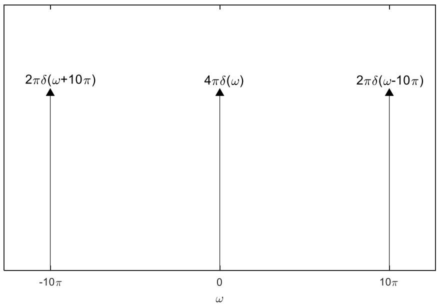
Properties that are used:

$$2\pi\delta(\omega) \xrightarrow{F^{-1}} 1$$

$$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \xrightarrow{F^{-1}} \cos(\omega_0 t)$$

$$s(t) = 2 + 2\cos(10\pi t)$$

Based on the magnitude response, S(jω) does not fit cleanly into one of the six frequency shapes (lowpass, highpass, bandpass, bandstop, notch, allpass).



c) Highpass signal.

$$R(j\omega) = \frac{j\omega}{4 + j2\omega} = \frac{1}{2} - \frac{2}{4 + j2\omega} = \frac{1}{2} - \frac{1}{2 + j\omega}$$

Properties:

$$1 \xrightarrow{F^{-1}} \delta(t)$$

$$\frac{1}{a + j\omega} \xrightarrow{F^{-1}} e^{-at}u(t)$$

$$r(t) = \frac{1}{2}\delta(t) - e^{-2t}u(t)$$

Alternate Solution:

$$j\omega X(j\omega) \xrightarrow{F^{-1}} \frac{dx(t)}{dt}$$

$$R(j\omega) = \frac{j\omega}{4 + j2\omega} = \frac{1}{2}\left(\frac{j\omega}{2 + j\omega}\right) \xrightarrow{F^{-1}} r(t) = \frac{1}{2}\frac{d}{dt}\left(e^{-2t}u(t)\right)$$

$$r(t) = \frac{1}{2}e^{-2t}\frac{d}{dt}u(t) + \frac{1}{2}u(t)\frac{d}{dt}e^{-2t} = \frac{1}{2}\delta(t) - e^{-2t}u(t)$$

d)

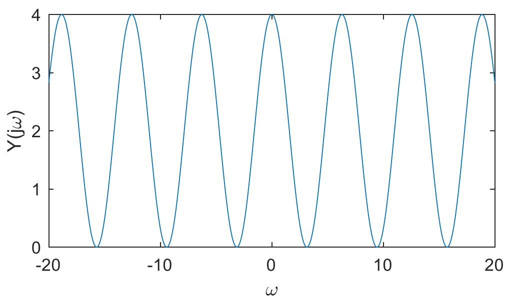
$$y(t) = \delta(t+1) + 2\delta(t) + \delta(t-1)$$

Properties:

$$\delta(t) \xrightarrow{F} 1$$

$$\delta(t - t_d) \xrightarrow{F} e^{-j\omega t_d}$$

$$Y(j\omega) = e^{j\omega} + 2 + e^{-j\omega} = 2 + 2\cos(\omega)$$



Based on the magnitude response, S(jω) does not fit cleanly into one of the six frequency shapes (lowpass, highpass, bandpass, bandstop, notch, allpass).

Problem 3

a)

$$x(t) = \frac{d}{dt}\left(\frac{10\sin(200\pi t)}{\pi t}\right)$$

Properties:

$$\frac{d^k x(t)}{dt} \xrightarrow{F} (j\omega)^k X(j\omega)$$

$$\frac{\sin(\omega_b t)}{\pi t} \xrightarrow{F} [u(\omega+\omega_b) - u(\omega-\omega_b)]$$

$$X(j\omega) = (j\omega)\left[10\left(u(\omega+200\pi) - u(\omega-200\pi)\right)\right] = j10\omega[u(\omega+200\pi) - u(\omega-200\pi)]$$

b)

$$x(t) = \frac{2\sin(400\pi t)}{\pi t}\cos(2000\pi t)$$

$$x(t)p(t) \rightleftharpoons \frac{1}{2\pi}X(j\omega) * P(j\omega)$$

$$\frac{\sin(\omega_b t)}{\pi t} \rightleftharpoons \left[u(\omega+\omega_b) - u(\omega-\omega_b)\right]$$

$$\cos(\omega_0 t) \rightleftharpoons \left[\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\right]$$

$$X(j\omega) = \frac{1}{2\pi}\left[u(\omega+400\pi) - u(\omega-400\pi)\right] * \left[\pi\delta(\omega+2000\pi) + \pi\delta(\omega-2000\pi)\right]$$
$$= u(\omega+400\pi+2000\pi) - u(\omega-400\pi+2000\pi) + u(\omega+400\pi-2000\pi)u(\omega-400\pi-2000\pi)$$
$$= u(\omega+2400\pi) - u(\omega+1600\pi) + u(\omega-1600\pi) - u(\omega-2400\pi)$$

c) This question introduces the impulse train, a.k.a. the Dirac comb. The signal introduced here is used for ideal sampling in which each Dirac delta models the instantaneous closing and opening of a gate to sample the amplitude (voltage) of a signal.

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-10n)$$

This signal has a fundamental period of Ts = 10 seconds, and the Fourier series coefficients can be calculated as follows:

$$a_k = \frac{1}{10}\int_{-5}^{5} \delta(t)e^{-j\frac{2\pi}{10}kt}\,dt = \frac{1}{10}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt} = \frac{1}{10}\sum_{k=-\infty}^{\infty} e^{j\frac{\pi}{5}kt}$$

$$e^{j\omega_0 t} \rightleftharpoons 2\pi\delta\left(\omega-\omega_0\right)$$

$$X(j\omega) = \frac{1}{10}\sum_{k=-\infty}^{\infty} 2\pi\delta\left(\omega-\frac{\pi}{5}k\right) = \frac{\pi}{5}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{\pi}{5}k\right)$$

So, an impulse train in the time domain is an impulse train in the frequency domain. That is, the process of sampling will create frequencies from $-\infty$ to $+\infty$.

We could also use the following transform pair in Table 11-2, which gives the same answer:

$$\sum_{n=-\infty}^{\infty}\delta(t-nT) \rightleftharpoons \frac{2\pi}{10}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi}{T}k\right)$$

The periodic impulse train in the time domain can be used to model an idealized view of sampling. When sampling the continuous-time signal $x(t)$ every T_s seconds, we ideally select the amplitude at each sampling time. This can be modeled mathematically in continuous time by multiplying $x(t)$ by an impulse train composed of continuous-time impulses (i.e. Dirac delta functionals) that occur at nT_s seconds where n is the sample index. Please the solution to homework problem 8.4 from fall 2017 at

http://users.ece.utexas.edu/~bevans/courses/signals/homework/fall2017/solution8.pdf

Problem 4

$x(t)$ is a real-valued signal, therefore:

$$x(t) = x^*(t)$$

$$x(t) \rightleftharpoons X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$x(t) \rightleftharpoons \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\,dt = \left(\int_{-\infty}^{\infty} x(t)e^{-(-j\omega t)}\,dt\right)^* = X^*(-j\omega)$$

$$x(t) = x^*(t) \rightleftharpoons X(j\omega) = X^*(-j\omega)$$

$$X(j\omega) = A(\omega) + jB(\omega), \qquad X^*(-j\omega) = A(-\omega) - jB(-\omega)$$

$$A(\omega) = A(-\omega), \qquad B(\omega) = -B(-\omega)$$

a)

$$\left|X(-j\omega)\right| = \sqrt{A^2(-\omega)+B^2(-\omega)} = \sqrt{A^2(\omega)+B^2(\omega)} = \left|X(j\omega)\right|$$

And this proves that magnitude is an even function.

b)

$$\angle X(-j\omega) = \tan^{-1}\left(\frac{B(-\omega)}{A(-\omega)}\right) = \tan^{-1}\left(\frac{-B(\omega)}{A(\omega)}\right) = -\tan^{-1}\left(\frac{B(\omega)}{A(\omega)}\right) = -\angle X(j\omega)$$

And this proves that phase is an odd function.