Solution Set for Homework #1 ***Version 4.0***

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PROBLEM 1

Prologue: This problem helps you to identify the points of interest in a sinusoidal signal and calculate the parameters of the waveform based on your observations. It relies on the definitions given in Sec. 2-3 of the *Signal Processing First* textbook.

Solution: To calculate A, consider the peak-to-peak amplitude (X_{pp}) of the waveform which is the difference between the maximum value and the minimum value:

X_{pp} = Maximum – Minimum = 4 – (–4) = 8. So, $\underline{A = X_{pp}/2 = 4}$.

To estimate the frequency, we would first estimate the period and the invert of the period. We could estimate a period in the plot by measuring the amount of time it takes to go from a peak to the next peak, or from one valley to the next. The plot has valleys at three different instances:

- t_0 = -2.4 ms,
- t_1 = -0.9 ms,
- t_2 = 0.6 ms.

One period has elapsed from t_0 to t_1 and two periods have elapsed from t_0 to t_2. The period (T) is about 1.5 ms long.

$T = t_1 - t_0 = -0.9 - (-2.4) = -0.9 + 2.4 = 1.5\text{ms}$ **OR** $T = \frac{t_2 - t_0}{2} = \frac{0.6-(-2.4)}{2} = 1.5\text{ms}$.

So, $\underline{\omega_0 = 2\pi/T = 1333.33\pi = 4188.79 \text{ rad/s or } f_0 = 667 \text{ Hz.}}$ <u>(Ground truth is 660 Hz.)</u>

We are left with one unknown, φ. We can pick any point in time to give us one equation in one unknown: $x(t) = A\cos(\omega_0 t + \varphi)$

- *Approach #1:* If we pick $t = 0$, then $x(0) = 4\cos(\varphi) \approx 3.25$ and $\varphi \approx \cos^{-1}(3.25/4) = 0.622$ rad (1)

Recall that cos(-φ) = cos(φ), so φ could be +0.622 rad or -0.622 rad. Also, any addition of φ and a multiple of 2π would also be a valid answer.

To decide between the values, consider the slope at $t = 0$, which is negative. (2)

$x(t) = A\cos(\omega_0 t + \varphi)$ => $x'(t) = -A\,\omega_0 \sin(\omega_0 t + \varphi)$ => $x'(0) = -A\,\omega_0 \sin(\varphi)$

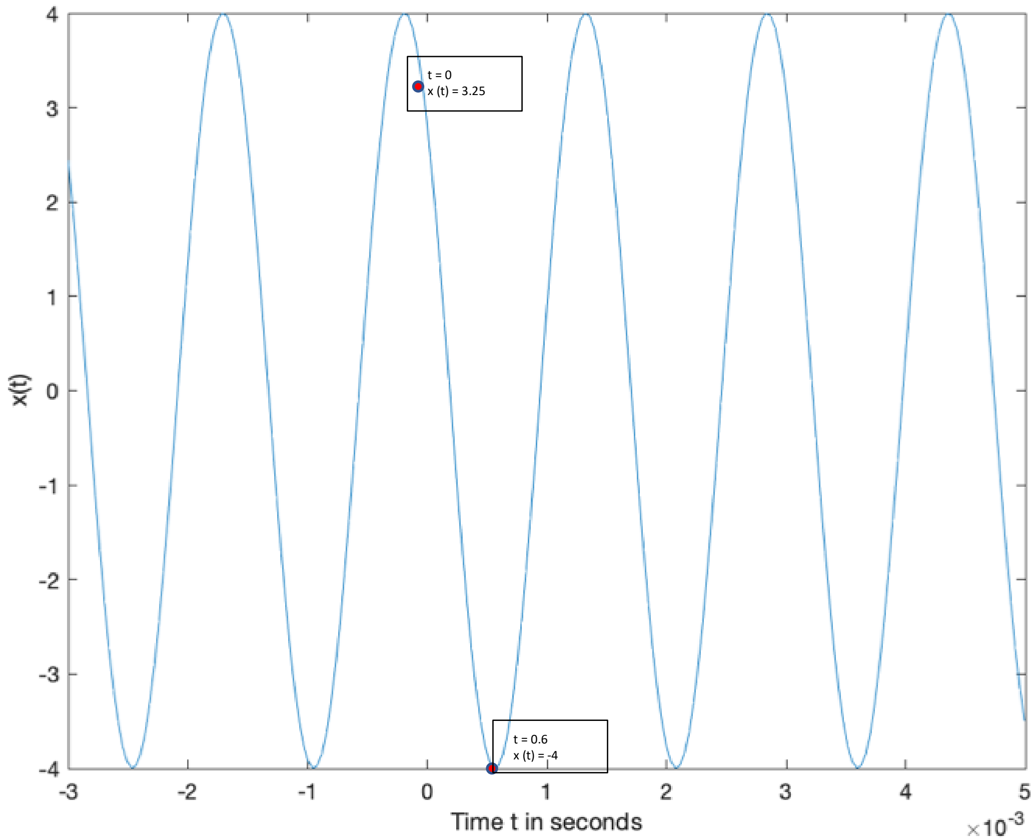
So, condition (2) implies that $x'(0) < 0$ => $\sin(\varphi) > 0$.

Out of the two values in (1), only $\underline{\varphi = 0.198\pi = 0.622 \text{ rad}}$ satisfies the condition.

Values used to <u>generate the plot</u> $A = 4$ $f_0 = 660$ Hz $\varphi = \pi/4 \approx 0.7854$ rad Note: 660 Hz is a C in the 4th octave on Western scale.

- *Approach #2.* Pick $t = 0.6$ ms where there is a valley. Then,

$x(0.6\text{ms}) = 4\cos(2\pi(0.6\text{ms}/1.5\text{ms}) + \varphi) = -4$ => $\cos(4\pi/5 + \varphi) = -1$ =>
$4\pi/5 + \varphi = \pi$ => $\underline{\varphi = \pi/5 = 0.6283 \text{ rad}}$.



Epilogue: There is nothing special about the time points considered for the solution. The points were chosen due to convenience, and the problem could have been done by taking other points as well. Additionally, the value of the phase φ is not unique. Using a ruler to measure points of interest will help give more accurate answers.

PROBLEM 2

Prologue: This problem helps you calculate the summation of several sinusoids having the same amplitude and frequency, but with different phases. By using the phasor addition rule, the answer can be derived.

Part (a)

$$x(t) = 120\cos(\omega_0 t) + 120\cos(\omega_0 t + 30^\circ) = 120\cos(\omega_0 t) + 120\cos(\omega_0 t + \pi/6)$$

Using the phasor addition rule from section 2-6.2 of the textbook:

$$Re\left\{120e^{j\omega_0 t} + 120e^{j\left(\omega_0 t+\frac{\pi}{6}\right)}\right\} = Re\left\{120e^{j\omega_0 t} + 120e^{j\omega_0 t}e^{\frac{j\pi}{6}}\right\}$$

$$= Re\left\{120\left(1 + e^{\frac{j\pi}{6}}\right)e^{j\omega_0 t}\right\} = Re\{Ae^{j\phi}e^{j\omega_0 t}\}$$

Using Euler's formula:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$120e^{\frac{j\pi}{6}} = 120\cos\left(\frac{\pi}{6}\right) + j120sin\left(\frac{\pi}{6}\right) = 60\sqrt{3} + j60$$

Therefore:

$$Ae^{j\phi} = 120\left(1 + e^{\frac{j\pi}{6}}\right) = 120 + 60\sqrt{3} + j60 = 223.923 + j60 = 231.82e^{j0.262}$$

So, A = 231.82 & φ = 0.262 rad

$$x(t) = Re\{231.82e^{j(\omega_0 t + 0.262)}\} = 231.82\cos(\omega_0 t + 0.262)$$

See the next page for the plot.

Part (b)

MATLAB Code:

```
w0 = 120*pi;
f0 = w0/(2*pi); %fundamental period in Hz
fs = 40*f0;     %Sampling frequency in Hz
Ts = 1/fs;      %Sampling time in seconds
t = -0.05:Ts:0.05;
x = 120*cos(w0*t)+120*cos(w0*t+pi/6);
plot(t,x);
xlabel('Time(s)');
ylabel('Amplitude');
figure;
xParta = 231.82 * cos(w0*t + 0.262);
plot(t, xParta);
```

T_0 is the fundamental period of $x(t)$

$\omega_0 = 2\pi/T_0 = 120\pi$

$T_0 = (2\pi)/(120\pi) = 1/60s = 0.0167s$

This range, e.g. $-0.05 \leq t \leq 0.05$, covers $T_1 = 0.05 - (-0.05) = 0.1s$ of the time scale

$m = T_1 / T_0 = 0.1 / 0.0167 = 6$, where m = 6 is the number of periods that are included in the plot.

The two plots are identical! See next page for the plot.

Part (c)

The answer is calculated in part a:

$$z(t) = 231.82\cos(\omega_0 t + 0.262)$$

Figure from Part (a)

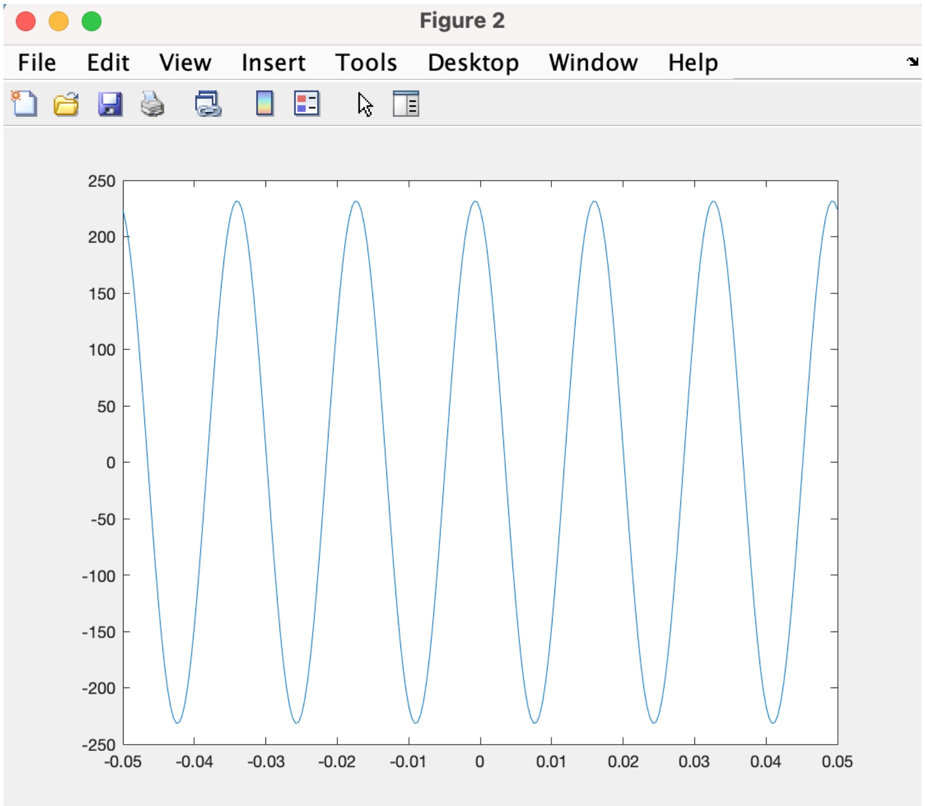
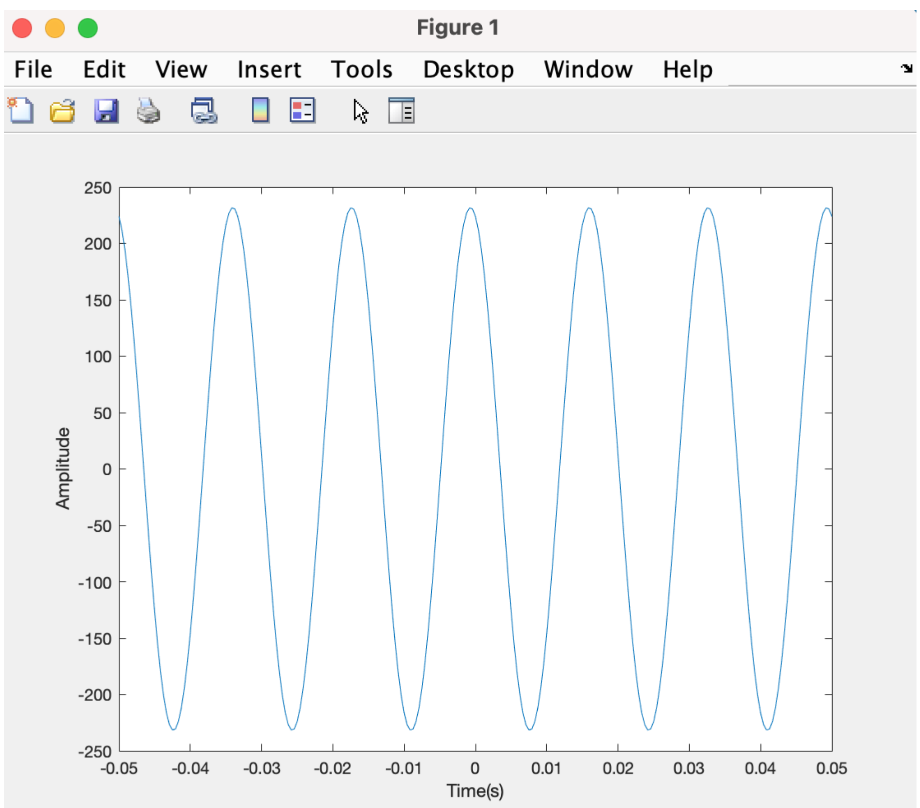


Figure from Part (b)



PROBLEM 3:

Prologue: This problem shows that a time shift leads to a phase shift. This principle is used in ranging systems that try to localize objects in an environment, such as ultrasound, sonar and radar. In those applications, the ranging system transmits a sinusoidal signal and then goes quiet. The signal will bounce off an object in the environment and return to the ranging system. The ranging system then computes the time delay between the transmitted and received signals to estimate the distance to the object by using the speed of propagation in the environment. The ranging system can compute the time delay in the time domain or the phase delay in the frequency domain. A common time domain method uses correlation, which is covered in EE 351K Probability and computed using convolution. We'll see convolution after midterm #1.

Here's the connection between time shift and phase shift:

$$x(t) = A\cos(2\pi f_0 t + \phi) = A\cos\big(2\pi f_0 (t - t_1)\big) = A\cos(2\pi f_0 t - 2\pi f_0 t_1)$$

Therefore, $\phi = -2\pi f_0 t_1 = -2\pi t_1 / T_0$

$T_0 = 8$s

Part (a)

$t_1 = -2$s
Therefore, the phase is equal to:

$$\phi = -\frac{2\pi(-2)}{8} = \frac{\pi}{2}$$

True

Part (b)
$t_1 = 3$s
Therefore, the phase is equal to:

$$\phi = -\frac{2\pi(3)}{8} = -\frac{3\pi}{4}$$

False

Part (c)
$t_1 = 7$s
Therefore, the phase is equal to:

$$\phi = -\frac{2\pi(7)}{8} = -\frac{7\pi}{4}$$

Due to property $\cos(x + 2\pi) = \cos(x)$. Each multiple of 2π corresponds to picking a different peak.

$$\phi = -\frac{7\pi}{4} + 2\pi = \frac{\pi}{4}$$

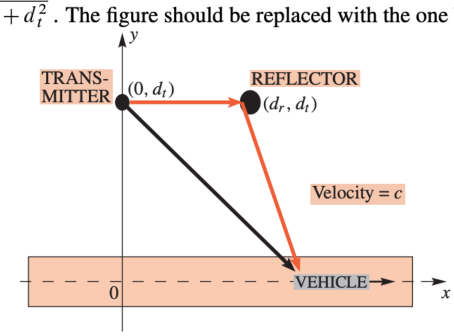
True

PROBLEM 4

Prologue: This problem shows how wireless reception can vary with the location of the receiving equipment (e.g. smart phone). In some locations, the receiving equipment will experience very weak reception, and strong reception in other locations. For the basic model of one reflector in this problem, the received amplitude strength is periodic with location on the horizontal axis. A takeaway is that by moving the receiving equipment, e.g. walking while on a videocall on the smart phone will provide a good connection on average.

Error in the textbook figure from the book errata

3. page 34*, Figure 2-21, The diagram of the original figure does not correspond to the equations given in the problem. The general formula for the distance off the reflector, d_2, is $d_2 = d_r + \sqrt{(x-d_r)^2 + d_t^2}$. The figure should be replaced with the one below:



Part (a)

$$t_1 = \frac{d_1}{c};\ d_1 = \sqrt{x^2 + d_t^2} = \sqrt{x^2 + 10^6}\ \text{m} \ \text{ and } c = 3 \times 10^8\ \text{m/s}$$

$$t_1 = \frac{\sqrt{x^2 + 10^6}}{3 \times 10^8}\ \text{s}$$

$$t_2 = \frac{d_2}{c}; d_2 = \sqrt{(x - d_r)^2 + d_t^2 + d_r} = \sqrt{(x - 55)^2 + 10^6} + 55\ \text{m}$$

$$t_2 = \frac{\sqrt{(x - 55)^2 + 10^6} + 55}{3 \times 10^8}\ \text{s}$$

Part (b)

time delay at $x = 0\text{m}$

$$t_1 = \frac{\sqrt{10^6}}{3 \times 10^8} = 3.3333 \times 10^{-6}\ \text{s}$$

$$t_2 = \frac{\sqrt{(-55)^2 + 10^6} + 55}{3 \times 10^8} = 3.5217 \times 10^{-6}\ \text{s}$$

$$s(t) = \cos(300 \times 10^6 \pi t)$$

$$r(t) = s(t - t_1) + s(t - t_2)$$

$$= \cos\left(300 \times 10^6 \pi (t - 3.3333 \times 10^{-6})\right) + \cos\left(300 \times 10^6 \pi (t - 3.5217 \times 10^{-6})\right)$$

$$= \cos(300 \times (10^6 \pi t - 1000\pi) + \cos(300 \times 10^6 \pi t - 1056.5\pi)$$

Using the phasor addition rule:

$$Re\{e^{j(300\times10^6\pi t)}e^{j(-1000\pi)} + e^{j(300\times10^6\pi t)}e^{j(-1056.5\pi)}\}$$
$$= Re\{e^{j(300\times10^6\pi t)}[e^{j(-1000\pi)} + e^{j(-1056.5\pi)}]\}$$
$$= Re\{Ae^{j\phi}e^{j\omega_0 t}\}$$

Using Euler's formula: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$$e^{j(-1000\pi)} = \cos(-1000\pi) + j\sin(-1000\pi) = 1 + j(0) = 1$$
$$e^{j(-1056.5\pi)}\cos(-1056.5\pi) + j\sin(-1056.5\pi)$$
$$= 0 + j(-1) = -1j$$
$$\phi = \tan^{-1}\left\{\frac{\sin(-1000\pi) + \sin(-1056.5\pi)}{\cos(-1000\pi) + \cos(-1056.5\pi)}\right\}$$
$$= \frac{-\pi}{4}$$

$$A = \sqrt{(1+0)^2 + (0+1)^2} = \sqrt{2}$$

$$f(t) = \sqrt{2}\cos\left(300\times10^6\pi t - \frac{\pi}{4}\right)$$

Part (c)

$r(t) = s(t-t_1) + s(t-t_2)$

$= \cos(300\times10^6\pi(t-t_1)) + \cos(300\times10^6\pi(t-t_2))$

A

$= \sqrt{[\cos(300\times10^6\pi t_1) + \cos(300\times10^6\pi t_2)]^2 + [(\sin(300\times10^6\pi t_1) + \sin(300\times10^6\pi t_2)]^2}$

when signal strength $= 0 \rightarrow A = 0$

$m = 300\times10^6\pi$

$[\cos(mt_1) + \cos(mt_2)]^2 + [\sin(mt_1) + \sin(mt_2)]^2 = 0$

$$\cos^2(mt_1) + \cos^2(mt_2) + 2\cos(mt_1)\cos(mt_2) + \sin^2(mt_1) + \sin^2(mt_2) + 2\sin(mt_1)\sin(mt_2) = 0$$

$\cos^2(x) + \sin^2(x) = 1$

$2 + 2\cos(mt_1)\cos(mt_2) + 2\sin(mt_1)\sin(mt_2) = 0$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$2 + 2\cos\big(m(t_1 - t_2)\big) = 0.$
$\cos\big(300 \times 10^6 \pi (t_1 - t_2)\big) = -1$
since the denominator of t_1 and t_2 is equal to 3×10^8

$$\cos\left(\pi\sqrt{x^2 + 10^6} - \pi\sqrt{(x-55)^2 + 10^6} + 55\pi\right) = -1$$
$$\cos\left(\pi\sqrt{x^2 + 10^6} - \pi\sqrt{(x-55)^2 + 10^6}\right) = 1$$
$$x^2 + 10^6 = (x-55)^2 + y^6$$
$$x^2 = x^2 + 55^2 - 110x \rightarrow 110x = 55^2 \rightarrow x = 27.5 \text{ m}$$

Part (d)

```
x = -100:0.1:100;
c = 3e8;                % speed of light in m/s
dr = 55;
dt = 1e6;
w0 = 300*10^6*pi;       % carrier frequency (150 MHz)
t1 = sqrt(x.*x+dt)/c;
t2 = (sqrt((x.*x-2*dr*x+dr^2)+dt)+dr)/c;  % expanded sqrt((x-dr)^2 + dt) term
s1 = cos(w0*t1)+cos(w0*t2);
s2 = sin(w0*t1)+sin(w0*t2);
a = sqrt(s1.*s1 + s2.*s2);
plot(x, a);
xlabel('Distance in meters');
ylabel('Amplitude Strength');
```