

**Solution Set for Homework #2 Version 2.0**

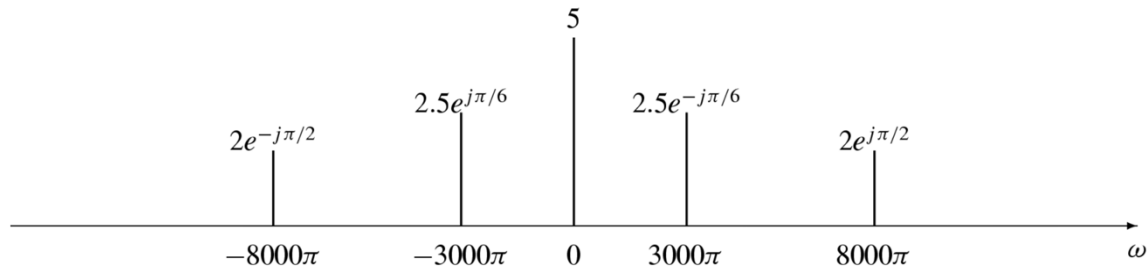
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**PROBLEM 1**

**Prologue:** This problem asks to generate the signal in the time domain from the signal's frequency domain representation (spectrum). The spectral and time representations give complementary views into the signal.

**Problem:** The spectrum follows



**Solution:** The spectrum indicates that the signal has frequency components of  $-8000\pi$  rad/s,  $-3000\pi$  rad/s,  $0$  rad/s,  $3000\pi$  rad/s and  $8000\pi$  rad/s. The strongest frequency component is at  $0$  rad/s because its magnitude (5) is the largest. The second strongest frequency components are at  $-3000\pi$  rad/s and  $3000\pi$  rad/s. The weakest frequency components are at  $-8000\pi$  rad/s and  $8000\pi$  rad/s.

**Part (a):** We can read off the spectral components to create the time-domain representation of the signal

$$\begin{aligned} x(t) &= 2e^{-j\frac{\pi}{2}}e^{-j2\pi(4000)t} + 2.5e^{j\frac{\pi}{6}}e^{-j2\pi(1500)t} + 5e^{j0t} + 2.5e^{-j\frac{\pi}{6}}e^{j2\pi(1500)t} + 2e^{j\frac{\pi}{2}}e^{j2\pi(4000)t} \\ &= 5 + \frac{5e^{j\frac{\pi}{6}}e^{-j2\pi(1500)t} + 5e^{-j\frac{\pi}{6}}e^{j2\pi(1500)t}}{2} + \frac{4e^{-j\frac{\pi}{2}}e^{-j2\pi(4000)t} + 4e^{j\frac{\pi}{2}}e^{j2\pi(4000)t}}{2} \\ &= 5 + 5 \cos\left(2\pi(1500)t - \frac{\pi}{6}\right) + 4 \cos\left(2\pi(4000)t + \frac{\pi}{2}\right) \end{aligned}$$

**Part (b):** The signal  $x(t)$  is periodic with period  $T$ , if  $x(t+T) = x(t)$  for all values of  $t$ :

$$\begin{aligned} x(t) &= 5 + 5 \cos\left(2\pi(1500)t - \frac{\pi}{6}\right) + 4 \cos\left(2\pi(4000)t + \frac{\pi}{2}\right) \\ x(t+T) &= 5 + 5 \cos\left(3000\pi t + 3000\pi T - \frac{\pi}{6}\right) + 4 \cos\left(8000\pi t + 8000\pi T + \frac{\pi}{2}\right) \end{aligned}$$

In order for  $x(t+T) = x(t)$  for all values of  $t$ , we can equate the first terms in  $x(t+T)$  and  $x(t)$ , equate the second terms in  $x(t+T)$  and  $x(t)$ , and equate the third terms in  $x(t+T)$  and  $x(t)$ , and find values of  $T$  that work for all three terms.

The first terms are already equal to each other (5). The second terms are equal when  $2\pi(1500)T = 2\pi k$  where  $k$  is an integer; i.e., when  $T$  is  $1/1500$ ,  $2/1500$ ,  $3/1500$ , etc. The third terms are equal when  $2\pi(4000)T = 2\pi m$  where  $m$  is an integer, i.e. when  $T$  is  $1/4000$ ,  $2/4000$ ,  $3/4000$ ,  $4/4000$ ,  $5/4000$ ,  $6/4000$ ,  $7/4000$ ,  $8/4000$ , etc. The common values for the period  $T$  in seconds among all three terms are  $1/500$ ,  $2/500$ ,  $3/500$ ,  $4/500$ , etc., which means that the fundamental period is 2 ms, and the fundamental frequency is 500 Hz.

Alternately, the fundamental frequency could be found by computing the greatest common divisor (gcd) of 1500 Hz and 4000 Hz:  $f_0 = \text{gcd}(1500 \text{ Hz}, 4000 \text{ Hz}) = 500 \text{ Hz}$  and  $T_0 = 1/f_0 = 1/(500 \text{ Hz}) = 2 \text{ ms}$ .

**Part (c):**  $\cos(\Theta) = \text{Re}\{e^{j\Theta}\}$ . From the inverse Euler’s formula,

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Therefore, sinusoids should be shown by both frequencies.

**PROBLEM 2**

**Prologue:** This problem concerns amplitude modulation which is the basis for AM/FM radio, Wi-Fi, cellular, cable modems, and other communication systems. Amplitude modulation can also be used to create audio effects.

**Modulation Property.** From this problem, you’ll be able to derive a fundamental principle called the Modulation Property. It says that multiplying a signal  $v(t)$  by  $\cos(2\pi f_c t)$  in the time domain will in the frequency domain shift the frequency content in  $v(t)$  by  $f_c$  and by  $-f_c$ , as shown in the solution. Modulation allows the conversion of an audio signal  $v(t)$  into a wireless electromagnetic radio frequency (RF) signal centered at  $f_c$  for long-range broadcast across the state and country, or a communication signal into the 2.4 GHz RF band for Wi-Fi. Example carrier frequencies: AM radio ~1 MHz, FM radio ~100 MHz, and Wi-Fi 2.4-2.499 GHz. (There are other Wi-Fi bands.) Audio signals have content between 20 Hz and 20 kHz, which are low frequencies compared to RF carrier frequencies. Audio signals don’t propagate very far; e.g. sound from outdoor concert venues might propagate a few kilometers. By converting the audio signal to an AM radio signal, we can broadcast it for thousands of kilometers.

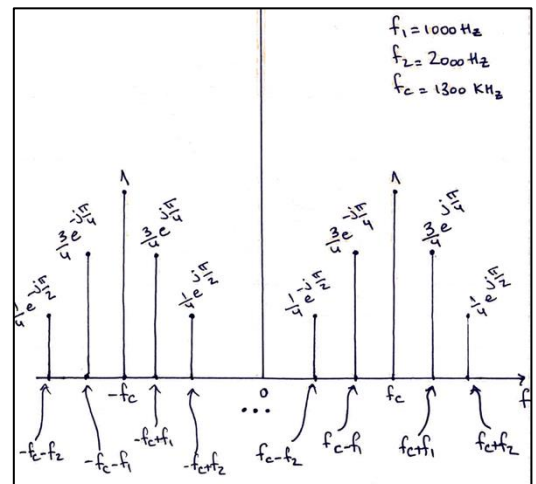
**Problem:** In AM radio, the transmitted signal (voice or music) is modulated by a sinusoid at the assigned broadcast frequency of the AM station. For example, the Zone Austin Sports Talk in Austin has a carrier frequency  $f_c$  of 1300 kHz. For example, if  $x(t)$  is the voice/music signal, then the transmitted signal would be:

$$y(t) = (x(t) + A) \cos(2\pi f_c t)$$

where  $A$  is a constant. ( $A$  is introduced to make the AM receiver design easier, in which case  $A$  must be chosen to be larger than the maximum value of  $v(t)$ .) Suppose that the signal that is to be transmitted is

$$x(t) = 3 \cos(2000\pi t + \pi/4) + \cos(4000\pi t + \pi/2)$$

Draw the spectrum for  $y(t)$  assuming a carrier frequency of 1300 kHz with  $A = 2$ . Hint: Substitute for  $x(t)$  and expand  $y(t)$  into a sum of cosine terms of three different frequencies.



**Solution:**

$$y(t) = \left[ 3 \cos\left(2\pi f_1 t + \frac{\pi}{4}\right) + \cos\left(2\pi f_2 t + \frac{\pi}{2}\right) + A \right] \cos(2\pi f_c t)$$

$$= 3 \cos\left(2\pi f_1 t + \frac{\pi}{4}\right) \cos(2\pi f_c t) + \cos\left(2\pi f_2 t + \frac{\pi}{2}\right) \cos(2\pi f_c t) + A \cos(2\pi f_c t)$$

First term (i) =  $3 \left[ \frac{1}{2} \left( e^{j\frac{\pi}{4}} e^{j2\pi f_1 t} + e^{-j\frac{\pi}{4}} e^{-j2\pi f_1 t} \right) \right] \left[ \frac{1}{2} \left( e^{j2\pi f_c t} + e^{-j2\pi f_c t} \right) \right]$

$$= \frac{3}{4} \left( e^{j\frac{\pi}{4}} e^{j2\pi(f_1+f_c)t} + e^{j\frac{\pi}{4}} e^{-j2\pi(f_c-f_1)t} + e^{-j\frac{\pi}{4}} e^{j2\pi(f_c-f_1)t} + e^{-j\frac{\pi}{4}} e^{-j2\pi(f_1+f_c)t} \right)$$

Second Term (ii) =  $\frac{1}{4} \left( e^{j\frac{\pi}{2}} e^{j2\pi(f_2+f_c)t} + e^{j\frac{\pi}{2}} e^{-j2\pi(f_c-f_2)t} + e^{-j\frac{\pi}{2}} e^{j2\pi(f_c-f_2)t} + e^{-j\frac{\pi}{2}} e^{-j2\pi(f_2+f_c)t} \right)$

Third Term (iii) =  $e^{j2\pi f_c t} + e^{-j2\pi f_c t}$



**PROBLEM 3**

**Prologue: Chirp Signals.** In a chirp signal, as mentioned in Section 3-8 of *Signal Processing First*, the principal frequency increases (or decreases) with time.

**Active sonar systems.** The transmitter sends out a “ping” as sound in form of a chirp and then receives sound. The time elapsed between the transmission and reception of the chirp indicates the roundtrip time experienced by the signal after bouncing off an object in the water and returning to the receiver. By receiving sounds in different directions using multiple microphones, the sonar can build a map of the objects in the water.

**Bats** use chirps for echolocation. Pipestrelle bats uses [chirps that sweep down from 70 to 45 kHz](#).

**Test & Measurement.** When one measures the response of a system to different frequencies, a time-consuming approach is to input a single sinusoid, measure the output, and repeat using many different frequencies. Instead, inputting a chirp can allow the measurement to be performed in one take.

**4G/5G cellular communication** systems periodically send a Zadoff-Chu chirp sequence to help measure the distortion in the electromagnetic propagation from transmitter to receiver.

**Prologue: Spectrograms.** For a periodic signal, Fourier series analysis tells us what frequencies are present and in what strengths. However, the analysis cannot tell us when in time a particular frequency occurs and at what strength.

Consider the Fourier series analysis of a square wave on lecture slide 3-11 with a fundamental period of 40ms. Fourier series analysis says that the square wave contains a frequency at 25 Hz with a strength of  $-j/\pi$ . However, the square wave has an amplitude of 0 from 20ms to 40ms, and hence, no frequency components are present in this time interval. The Fourier series analysis computes the *average* strength of a frequency component over the fundamental period.

A spectrogram can be used to analyze both periodic and non-periodic signals in the time and frequency domains simultaneously. The spectrogram indicates when in time a particular frequency occurs and at what strength.

**Problem:** This problem analyzes the chirp signal, which is used in sonar and radar systems, indoor positioning, test and measurement, and 4G cellular communications. The chirp signal is a sinusoid whose principal frequency content increases (or decreases) over time. A chirp signal has the form

$$c(t) = \cos(\theta(t)) \quad \text{where } \theta(t) = 2\pi(f_0 + \frac{1}{2}f_{\text{step}}t)t = 2\pi f_0 t + \pi f_{\text{step}} t^2$$

The principal frequency is  $f_0$  when  $t = 0$  and then changes over time at a rate of  $f_{\text{step}}$  in units of Hz/s. The principal frequency of a sinusoid at a given point in time is called the *instantaneous frequency*, and it is defined as  $d\theta(t)/dt$  in units of rad/s. Here,  $d\theta(t)/dt = 2\pi f_0 + 2\pi f_{\text{step}}t = 2\pi(f_0 + f_{\text{step}}t)$ .

**Part (a):** MATLAB code to generate the chirp signal that lasts 10s with  $f_0 = 20$  Hz and  $f_{\text{step}} = 420$  Hz/s. Use sampling rate  $f_s$  of 44100 Hz. The chirp will sweep through the [frequencies of the keys on an 88-key piano](#).

```
time = 10;           % length of time in seconds
f0 = 20;            % specify starting principal frequency
fstep = 420;       % specify frequency slope
fs = 44100;        % sampling rate
Ts = 1/fs;         % sampling time: time interval between samples
t = 0 : Ts : time; % create a time vector
theta = 2*pi*(f0+0.5*fstep*t).*t;
y = cos(theta);    % create chirp waveform
```

**Part (b):** The chirp signal can be played as an audio signal in MATLAB:

```
sound(y, fs);      % play back chirp signal
```

The chirp linearly sweeps frequencies from 20 to 4220 Hz, and sounds like a note increasing in “pitch” over time. The sweep spans [8 octaves of A notes on a Western scale](#): 27.5, 55, 110, 220, 440, 880, 1760, and 3520 Hz.

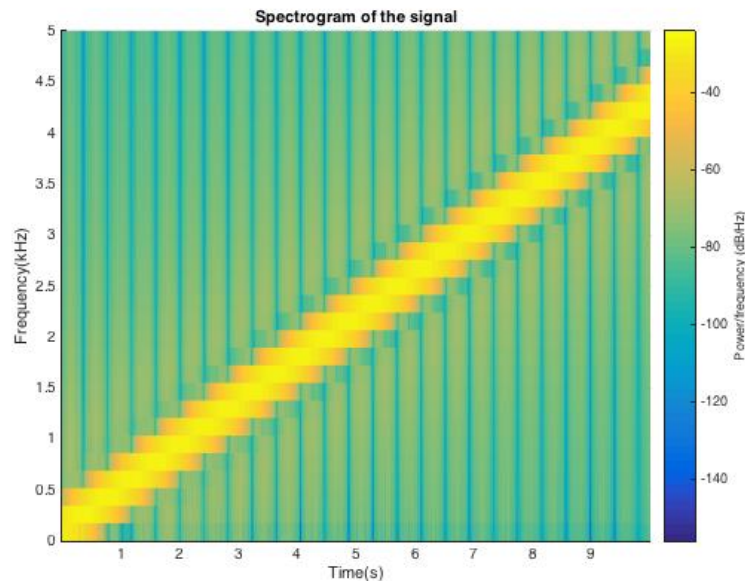
**Part (c):** We can visualize the variation of the principal frequency over time by using a time-frequency representation. The time-frequency representation below is called a spectrogram.

The spectrogram for the chirp signal (next page) shows the principal frequency changing over time with a linear slope. The principal frequency is initially at 20 Hz and increases linearly to 4220 Hz. The principal frequency has the highest magnitude at every instant of time throughout the entire duration of the chirp signal, which is shown in yellow.

As described on lecture slides 4-8 and 4-9, a spectrogram takes the first  $N_{\text{win}}$  samples of the signal, weights the values (using a rectangular pulse by default), computes the Fourier series coefficients, and plots the magnitude of the Fourier series coefficients vertically. The spectrogram then shifts the time signal to the right and repeats the previous steps using a block of the next  $N_{\text{win}}$  samples. The frequency resolution of the spectrogram is  $f_s / N_{\text{win}}$ .

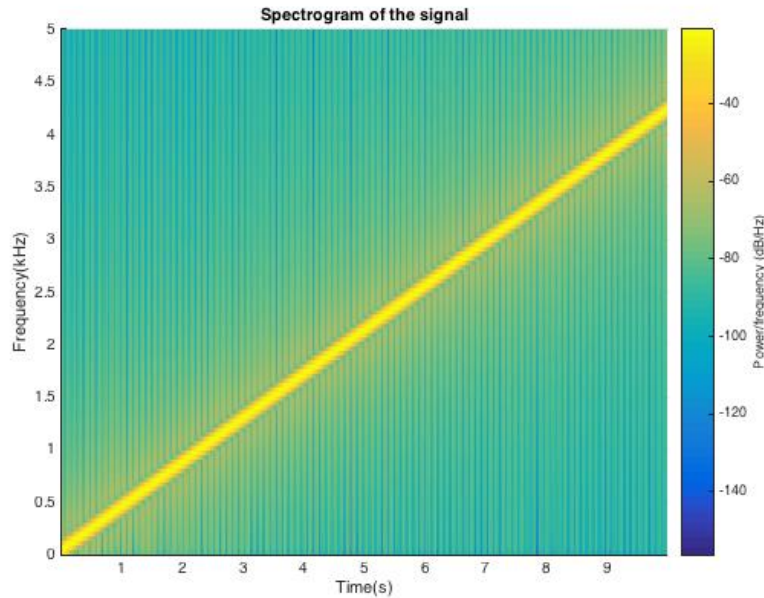
In the code below, a Hamming window is used to weight the values in each block of samples.

```
figure;
spectrogram(y, hamming(256), 128, 256, fs, 'yaxis')
ylim([0,5]);
title('Spectrogram of the signal');
ylabel('Frequency (kHz)');
xlabel('Time (s)');
```



Although not asked, we can increase the frequency resolution by increasing the block size:

```
figure;
spectrogram(y, hamming(1024), 512, 1024, fs, 'yaxis')
ylim([0,5]);
title('Spectrogram of the signal');
ylabel('Frequency (kHz)');
xlabel('Time (s)');
```



#### **PROBLEM 4**

**Prologue:** This problem explores various audio effects by applying nonlinear operations to a sinusoidal signal. It also explores what happens to the frequency component of a sinusoidal signal due to nonlinear operations. This is a common approach in analyzing how a nonlinear system (e.g. a diode or transistor) responds to different frequencies. Among the nonlinear operations, parts (a), (b) and (c) can be used in amplitude demodulation (see the Prologue for Problem 2). The nonlinear operation in part (d) is a type of frequency modulation.

**Problem:** Consider the signal  $x(t) = \cos(2\pi f_0 t)$  where  $f_0 = 440$  Hz which an ‘A’ note on the Western scale.

Write MATLAB code to implement the following audio effects. Play each audio signal over  $0 \leq t \leq 5$  and use a sampling rate of 8000 Hz:

- (a)  $y(t) = |x(t)|$
- (b)  $y(t) = x^2(t)$
- (c)  $y(t) = x^4(t)$
- (d)  $y(t) = \cos(\pi x(t))$

For each part, give a mathematical analysis to determine what frequencies are present in  $y(t)$  and play  $y(t)$  as an audio signal and describe what you hear vs.  $x(t)$ .

**Solution:** MATLAB code for generating and playing  $x(t)$  and output signals for parts (a)-(d):

```
f0 = 440;
fs = 8000;
Ts = 1/fs;
t = 0:Ts:5;
x = cos(2*pi*f0*t);
y_a = abs(x);
y_b = x.^2;
y_c = x.^4;
y_d = cos(pi*x);

figure(1)
spectrogram(x, hamming(256), 128, 256, fs, 'yaxis')
sound(x, fs)
pause(8);
```

```

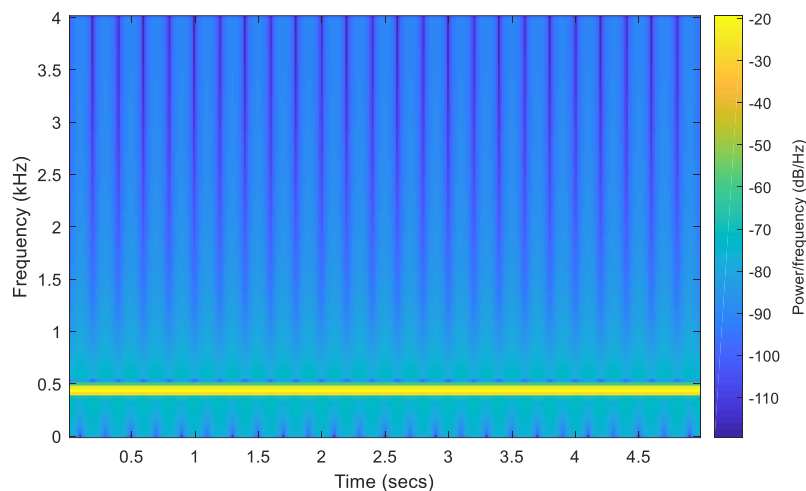
figure(2)
spectrogram(y_a, hamming(256), 128, 256, fs, 'yaxis')
soundsc(y_a, fs)
pause(8);

figure(3)
spectrogram(y_b, hamming(256), 128, 256, fs, 'yaxis')
soundsc(y_b, fs)
pause(8);

figure(4)
spectrogram(y_c, hamming(256), 128, 256, fs, 'yaxis')
soundsc(y_c, fs)
pause(8);

figure(5)
spectrogram(y_d, hamming(256), 128, 256, fs, 'yaxis')
soundsc(y_d, fs)

```



$x(t)$  only contains the frequencies  $\pm 440$  Hz, and the spectrogram shows this.

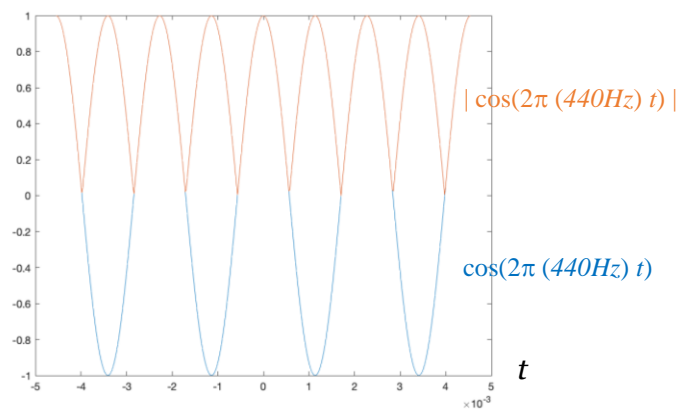
**Note:** We're sampling a continuous-time signal at a sampling rate of  $f_s = 8000$  Hz. From the sampling theorem,  $f_s > 2 f_{\max}$  or by dividing each side by 2,  $f_{\max} < (1/2) f_s$ . By sampling, we can only capture continuous-time frequencies up to  $(1/2) f_s$ , or 4000 Hz. We won't be able to see harmonics at or above 4000 Hz in the spectrogram.

**Part (a):** The fundamental period of  $y(t)$  is half that of  $x(t)$ . Here are plots of four/eight fundamental periods of  $x(t)/y(t)$ :

In order to be periodic,  $y(t) = y(t+T_0)$  and  $T_0$  is the fundamental period in seconds where  $T_0 = 1/880$  in seconds.

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{j \frac{2\pi k}{T_0} t}$$

$$a_k = \frac{1}{T_0} \int_{t=0}^{T_0} y(t) e^{-j \frac{2\pi k}{T_0} t} dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t) e^{-j \frac{2\pi k}{T_0} t} dt$$

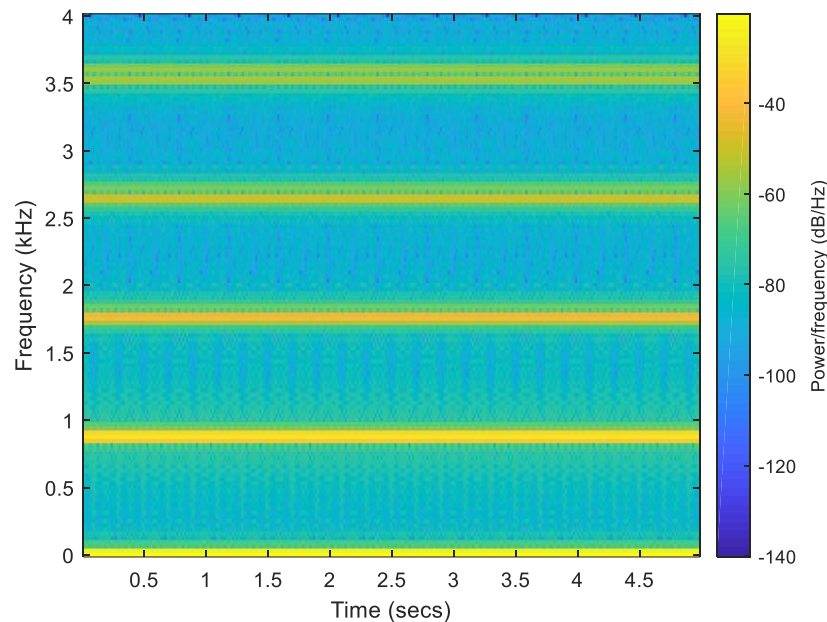


$$a_0 = 880 \int_{-\frac{1}{1760}}^{\frac{1}{1760}} \cos(2\pi(440)t) dt = \frac{880}{2\pi(440)} \left( \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) \right) = \frac{2}{\pi}$$

And

$$\begin{aligned} a_k &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t) e^{-j2\pi k f_0 t} dt = 880 \int_{-\frac{1}{1760}}^{\frac{1}{1760}} |\cos(2\pi(440)t)| e^{-j2\pi k(880)t} dt = 880 \int_{-\frac{1}{1760}}^{\frac{1}{1760}} \cos(2\pi(440)t) e^{-j2\pi k(880)t} dt = \\ &= 880 \int_{-\frac{1}{1760}}^{\frac{1}{1760}} \frac{e^{j2\pi(440)t} + e^{-j2\pi(440)t}}{2} e^{-j2\pi k(880)t} dt = 880 \int_{-\frac{1}{1760}}^{\frac{1}{1760}} \frac{e^{-j2\pi(880k-440)t} + e^{-j2\pi(880k+440)t}}{2} dt \\ &= \frac{880}{2} \left[ -\frac{1}{j2\pi(880k-440)} (e^{-j2\pi(0.5k-0.25)} - e^{j2\pi(0.5k-0.25)}) - \frac{1}{j2\pi(880k+440)} (e^{-j2\pi(0.5k+0.25)} - e^{j2\pi(0.5k+0.25)}) \right] \\ &= e^{-j\pi k} \left[ -\frac{1}{2\pi(2k-1)} + \frac{1}{2\pi(2k+1)} \right] + e^{j\pi k} \left[ -\frac{1}{2\pi(2k-1)} + \frac{1}{2\pi(2k+1)} \right] = \frac{2 \cos(\pi k)}{\pi(1-4k^2)} \end{aligned}$$

So,  $y(t)$  contains an infinite number of even harmonics of 440 Hz, which correspond to frequencies of  $\pm 880k$  where  $k$  is an integer value. As expected, we can hear a higher frequency sound. The sound feels richer as several harmonics are present (less 'thin' than a pure cosine). The Sampling Theorem says to choose the sampling rate so that  $f_s > 2 f_{\max}$  and hence  $f_{\max} < \frac{1}{2} f_s$ . Sampling at 8 kHz will only capture harmonic frequencies below 4 kHz.



**Part (b):**

$$y(t) = \cos^2(2\pi 440t)$$

$$y(t) = \frac{1}{2} + \frac{1}{2} \cos(2\pi 880t)$$

(Using the trig. identity  $\cos(2\theta) = 2\cos^2(\theta) - 1$ )

$y(t)$  contains 0 Hz and  $\pm 880$  Hz, which are even harmonics of 440 Hz up to the 2nd harmonic. Alternatively,

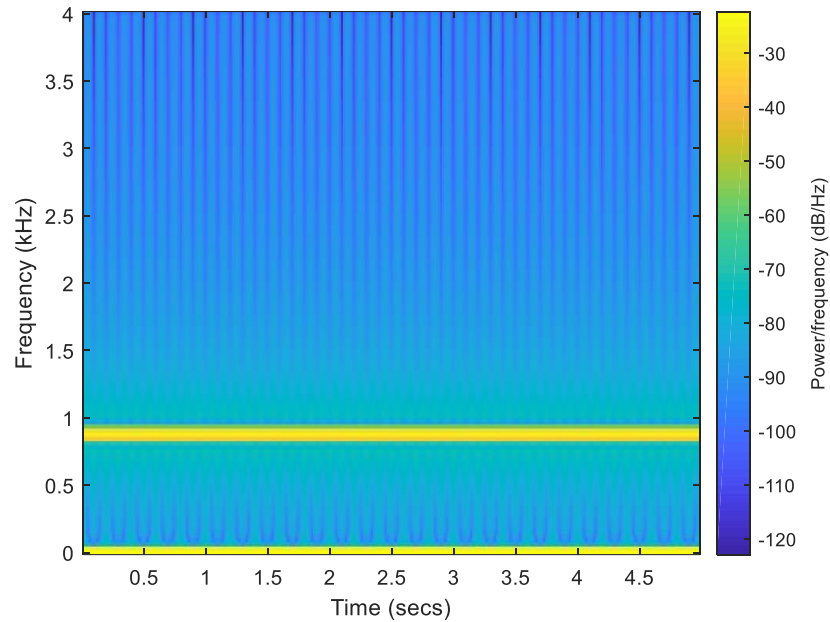
$$y(t) = \cos^2(2\pi 440t)$$

$$\begin{aligned} y(t) &= \frac{(e^{j2\pi 440t} + e^{-j2\pi 440t})^2}{4} \\ &= \frac{e^{j2\pi 880t} + e^{-j2\pi 880t} + 2}{4} = \frac{1}{2} + \frac{\cos(2\pi 880t)}{2} \end{aligned}$$



(Using the binomial expansion for  $(a+b)^2 = a^2+b^2+2ab$ )

It sounds ‘thin’, which is because just one frequency is audible in the waveform, while in part (a), many harmonics could be also heard.



**Part (c):**

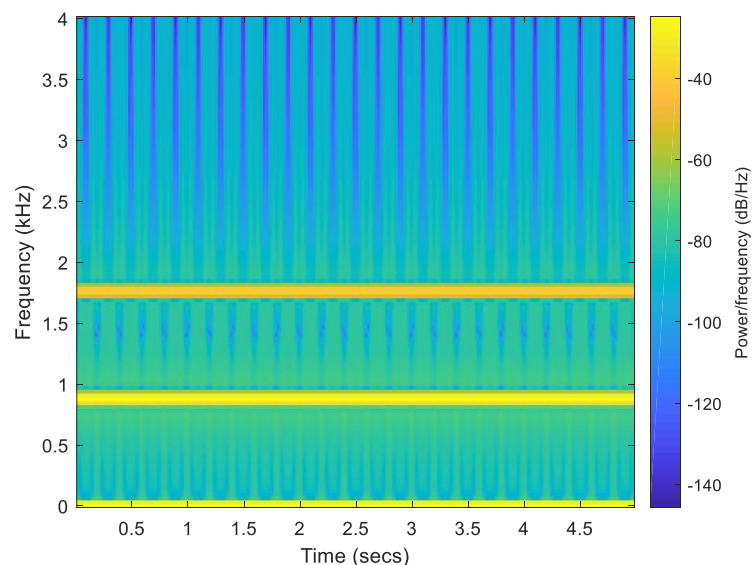
$$y(t) = x^4(t) = \cos^4(2\rho f_0 t) = \left( \frac{e^{j2\rho f_0 t} + e^{-j2\rho f_0 t}}{2} \right)^4 = \frac{e^{j8\rho f_0 t} + e^{-j8\rho f_0 t} + 4e^{j4\rho f_0 t} + 4e^{-j4\rho f_0 t} + 6}{16}$$

$$= 0.375 + 0.5\cos(4\rho f_0 t) + 0.125\cos(8\rho f_0 t) = 0.375 + 0.5\cos(2\rho(880)t) + 0.125\cos(2\rho(1760)t)$$

(Using the binomial expansion for  $(a + b)^4 = a^4 + b^4 + 4 a^3 b + 4 a b^3 + 6a^2b^2$ )

So,  $y(t)$  contains the frequencies 0 Hz,  $\pm 880$ Hz and  $\pm 1760$ Hz, which are the second and fourth harmonics of 440 Hz.

This sounds as if it has a frequency between the signals in parts (a) & (b). The sound is more ‘thin’ than part (a), while it is not pure as audio in part b, and the second harmonic frequency of 880 Hz (i.e.  $\pm 1760$ Hz) can be heard.





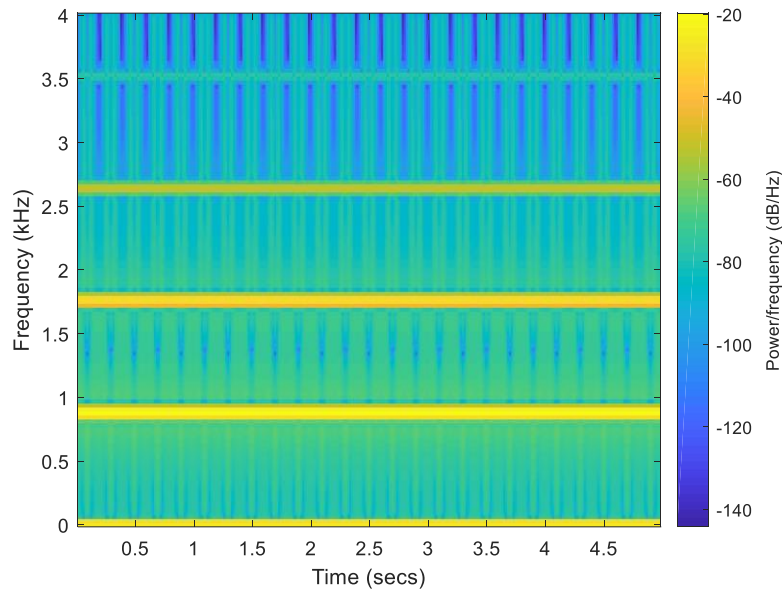
**Part (d):** Using Taylor series expansion:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\cos(\pi \cos(2\pi f_0 t)) = 1 - \frac{(\cos(2\pi f_0 t))^2}{2!} + \frac{(\cos(2\pi f_0 t))^4}{4!} - \frac{(\cos(2\pi f_0 t))^6}{6!} + \frac{(\cos(2\pi f_0 t))^8}{8!} - \dots$$

As shown in the above formula,  $y(t)$  includes  $(\cos(\pi \cos(2\pi f_0 t)))^2$ , and as shown in part (b) and (c),  $y(t) = \cos^n(2\pi f_0 t)$  will have odd harmonics up to the  $n$ th harmonic if  $n$  is odd, and even harmonics up to the  $n$ th harmonic if  $n$  is even including a zero-frequency component.

Compared to parts (b) and (c), the audio signal has additional harmonics.



**Epilogue:** Harmonics of a fundamental frequency give the note at the fundamental frequency a richer sound. When instruments play a note, the instrument creates harmonics, and the harmonics generated can help one identify which instrument it is.

From analyzing  $\cos^2(2\pi f_0 t)$  and  $\cos^4(2\pi f_0 t)$ , we can infer that  $\cos^n(2\pi f_0 t)$  has even harmonics of  $f_0$  up to the  $n$ th harmonic as well as a DC term when  $n$  is even. When  $n$  is odd, we can start with  $\cos^3(2\pi f_0 t)$  which has frequencies  $-3f_0$ ,  $-f_0$ ,  $f_0$  and  $3f_0$  from lecture slide 3-9. More generally,  $\cos^n(2\pi f_0 t)$  will have odd harmonics of  $f_0$  up to the  $n$ th harmonic if  $n$  is odd, and even harmonics of  $f_0$  up to the  $n$ th harmonic plus an DC term if  $n$  is even.