Solution Set for Homework #3 on Fourier Series and Sampling

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PROBLEM 1: FOURIER ANALYSIS AND SYNTHESIS

Prologue: The purpose of this problem is to use properties of the continuous-time Fourier series in computing the Fourier series coefficients. Throughout the remainder of the course, we'll be using properties of continuous-time Fourier transforms and other transforms to simplify the computation of the transform.

Problem: *Signal Processing First*, problem P-3.14, page 67. The problem gives an example of a signal $x(t)$ that has period T_0 and another signal $y(t) = \frac{d}{dt} x(t)$. The Fourier series coefficients b_k for $y(t)$ can be computed from the Fourier series coefficients a_k for $x(t)$ using $b_k = (j\, k\, \omega_0)\, a_k$ where $\omega_0 = 2\,\pi\, f_0$.

Solution for part (a): Here are two different solutions for $y(t) = A\, x(t)$.

Solution #1 for part (a)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t}$$

Let $y(t) = A\, x(t)$:

$$y(t) = A\left(\sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t}\right)$$

$$y(t) = \sum_{k=-\infty}^{+\infty} A\, a_k\, e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{+\infty} (A\, a_k\,)\, e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{+\infty} b_k\, e^{jk\omega_0 t}$$

$$b_k = A\, a_k$$

Solution #2 for part (a)

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$b_k = \frac{1}{T_0}\int_0^{T_0} y(t)\, e^{-jk\omega_0 t}\, dt$$

Let $y(t) = A\, x(t)$:

$$b_k = \int_0^{T_0} A\, x(t)\, e^{-jk\omega_0 t}\, dt$$

$$b_k = A\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$b_k = A\, a_k$$

When scaling any signal in amplitude, the Fourier Series coefficients are scaled by the same amount.

Solution for part (b): Here are two different solutions for $y(t) = A\,x(t - t_d)$.

Solution #1 for part (b)

Let $y(t) = x(t - t_d)$:

$$x(t - t_d) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0(t - t_d)}$$

$$x(t - t_d) = \sum_{k=-\infty}^{+\infty} a_k\, e^{-jk\omega_0 t_d}\, e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{+\infty} b_k\, e^{jk\omega_0 t}$$

$$b_k = e^{-jk\omega_0 t_d}\, a_k$$

Solution #2 for part (b)

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$b_k = \frac{1}{T_0}\int_0^{T_0} y(t)\, e^{-jk\omega_0 t}\, dt$$

Let $y(t) = x(t - t_d)$:

$$b_k = \int_0^{T_0} x(t - t_d)\, e^{-jk\omega_0 t}\, dt$$

Using a substitution of variables with $\lambda = t - t_d$ and $d\lambda = dt$. The limits of integration $t \to 0$ becomes $\lambda \to -t_d$ and $t \to T_0$ becomes $\lambda \to T_0 - t_d$,

$$b_k = \int_0^{T_0} x(\lambda)\, e^{-jk\omega_0(\lambda + t_d)}\, dt$$

$$b_k = \int_{-t_d}^{T_0 - t_d} x(\lambda)\, e^{-jk\omega_0 t_d} e^{-jk\omega_0 \lambda}\, d\lambda$$

$$b_k = e^{-jk\omega_0 t_d}\int_{-t_d}^{T_0 - t_d} x(\lambda)\, e^{-jk\omega_0 \lambda}\, d\lambda$$

$$b_k = e^{-jk\omega_0 t_d}\, a_k$$

When delaying a signal, the Fourier Series coefficients are multiplied by $e^{-jk\omega_0 t_d}$. This is another example of a shift in time causing shift in phase.

(c) $y(t) = 2x\left(t - \frac{1}{4}T_0\right)$

Using the conclusion derived in parts (a) and (b) with $A = 2$ and $t_d = ¼\, T_0$,

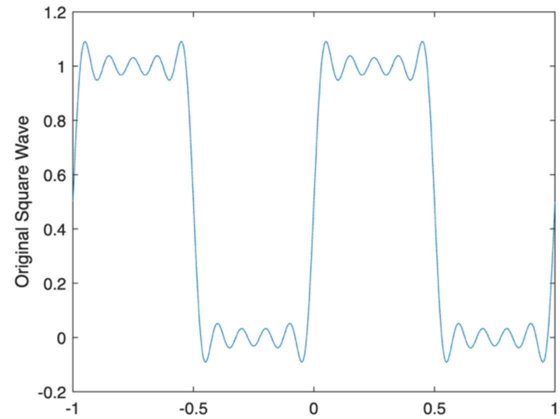
$$b_k = 2\, e^{-jk\omega_0\left(\frac{1}{4}T_0\right)} a_k$$

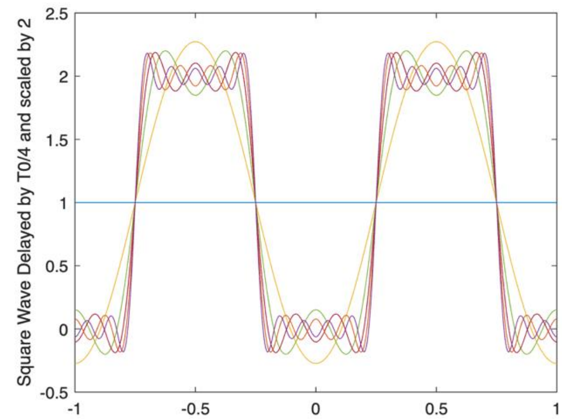
Given $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$,

$$b_k = 2\, e^{-jk\frac{\pi}{2}} a_k$$

(d) Below, the plots of $x(t)$ and $y(t)$ are plotted for two periods to better show the shift in time $y(t) = 2\,x\left(t - \frac{1}{4}T_0\right)$. Note the doubling in amplitude for $y(t)$.

```
% Fourier synthesis for square wave
% Prof. Brian L. Evans
% The University of Texas at Austin
% Written in Fall 2017
% Version 2.0
%
% Fourier series coefficients ak for a square
% wave with period T0 that is
% 1 for 0 <= t < T0/2
% 0 for T0/2 <= t < T0
%
% Derivation is in Sec. 3-6.1 in Signal
% Processing First (2003) on pages 52-53
% Pick a value for the period of x(t)
T0 = 1;
f0 = 1 / T0;
% Pick number of terms for Fourier synthesis
N = 10;
fmax = N * f0;
% Define a sampling rate for plotting
fs = 24 * fmax;
Ts = 1 / fs;
% Define samples in time for one period
%t = -0.5*T0 : Ts : 0.5*T0;
t = -T0 : Ts : T0;
% First Fourier synthesis term
a0 = 0.5;
b0 = 2*a0;
x = a0 * ones(1, length(t));
y = b0 * ones(1, length(t));
figure;
plot(t, y);
ylabel('Square Wave Delayed by T0/4 and scaled by 2')
hold on;
% Generate each pair of synthesis terms
for k = 1 : N
    % Define Fourier coefficients at k and -k
    akpos = (1 - (-1)^k) / (j*2*pi*k);
    akneg = (1 - (-1)^(-k)) / (j*2*pi*(-k));
    bkpos = 2*(exp(-j*2*pi*k*(1/4)*T0))*akpos;
    bkneg = 2*(exp(-j*2*pi*(-k)*(1/4)*T0))*akneg;
    theta = j*2*pi*k*f0*t;
    x = x + akpos * exp(theta) + akneg * exp(-theta);
    y = y + bkpos * exp(theta) + bkneg * exp(-theta);
    % Plot Fourier synthesis for indices -k ... k
    plot(t, y);
    pause(0.5);
end
hold off;
figure;
plot(t, x);
ylabel('Original Square Wave')
```





PROBLEM 2: SAMPLING

Prolog: Periodicity is a bit different for discrete-time signals than continuous-time signals because the discrete-time domain is on an integer grid whereas the continuous-time domain is on a real number line.

Problem: *Signal Processing First*, problem P-4.2, page 96, with an additional part (d).

$$x(t) = 7\sin(11\pi t) = 7\cos\left(11\pi t - \frac{\pi}{2}\right)$$

In the continuous-time domain, the fundamental period is (2/11) seconds:

$$\omega_0 = 11\pi \frac{\text{rad}}{\text{s}}$$

$$f_0 = \frac{11\pi}{2\pi} = 5.5 \text{ Hz}$$

$$T_0 = \frac{2}{11} s$$

$$\emptyset = -\frac{\pi}{2} \text{ rad}$$

(a) $\hat{w}_0 = 2\pi \frac{f_0}{f_s} = 2\pi \frac{5.5 \text{ Hz}}{10 \text{ Hz}} = \frac{11}{10}\pi \frac{\text{rad}}{\text{sample}}$

Due to sampling at f_s = 10 Hz, $x[n] = x(n\,T_s) = x\left(\frac{n}{f_s}\right)$:

$$x[n] = 7\cos\left(\frac{11\pi}{10}n - \frac{\pi}{2}\right)$$

$$= 7\cos\left(\frac{11\pi}{10}n - 2\pi n - \frac{\pi}{2}\right)$$

$$= 7\cos\left(-\frac{9\pi}{10}n - \frac{\pi}{2}\right)$$

$$= 7\cos\left(\frac{9\pi}{10}n + \frac{\pi}{2}\right)$$

A = 7, $\varphi = \frac{\pi}{2}$ rad

(b) $\hat{w}_0 = 2\pi \frac{5.5Hz}{5Hz} = \frac{11}{5}\pi \frac{\text{rad}}{\text{sample}}$

Due to sampling at f_s = 5 Hz, $x[n] = x(n\,T_s) = x\left(\frac{n}{f_s}\right)$:

$$x[n] = 7\cos\left(\frac{11\pi}{5}n - \frac{\pi}{2}\right)$$

This signal is undersampled, because $f_0 > f_s / 2$. The following equation shows the effect of aliasing (but not related to folding) caused by the undersampling:

$$x[n] = 7\cos\left(\frac{11\pi}{5}n - \frac{\pi}{2}\right) = 7\cos\left(\frac{11\pi}{5}n - 2\pi n - \frac{\pi}{2}\right) = 7\cos\left(\frac{\pi}{5}n - \frac{\pi}{2}\right)$$

A = 7, $\varphi = -\frac{\pi}{2}$ rad

(c) $\widehat{w}_0 = 2\pi \frac{5.5\, Hz}{15\, Hz} = \frac{11}{15}\pi \frac{\text{rad}}{\text{sample}}$

This signal is 15/11 times oversampled because $f_0 < f_s / 2$

$$x[n] = 7\cos\left(\frac{11\pi}{15}n - \frac{\pi}{2}\right)$$

A = 7, $\varphi = -\frac{\pi}{2}$ rad

(d) As shown at the beginning of this problem's solution:

$$f_0 = \frac{11\pi}{2\pi} = 5.5 \text{ Hz} \text{ and } T_0 = \frac{2}{11}s$$

According to the hint that is provided for this solution, which comes from Handout D on Discrete-Time Periodicity, $x[n]$ is periodic with a discrete-time period of N_0 samples if $x[n] = x[n + N_0]$ for all possible integer values of N_0 .

$$x[n + N_0] = 7\cos\left(\frac{11\pi}{15}(n + N_0) - \frac{\pi}{2}\right)$$

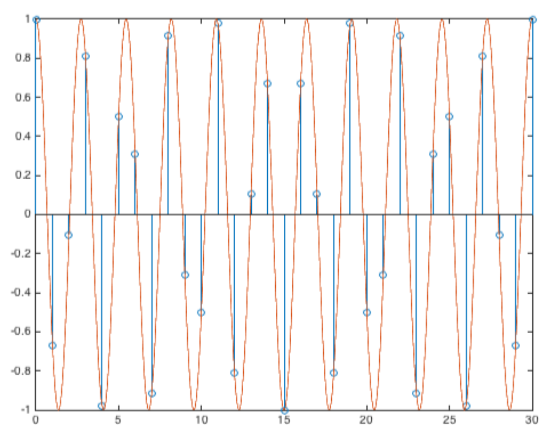
$$= 7\cos\left(2\pi\frac{11}{30}n + 2\pi\frac{11}{30}N_0 - \frac{\pi}{2}\right)$$

$$= 7\cos\left(2\pi\frac{11}{30}n - \frac{\pi}{2}\right)$$

Because 11 and 30 are relatively prime, the smallest possible positive integer for N_0 is 30 samples. Therefore, the fundamental period of $x[n]$ is 30 samples. Those 30 samples contain 11 continuous-time periods, which corresponds to 2.67 samples in each continuous-time period.

Although not required, here's a way to visualize differences in periodicity by superimposing plots of $x(t)$ and $x[n]$. In $x[n]$, the amplitude of 1 at $n = 0$ does not repeat until $n = 30$.

```
fs = 15;
Ts = 1/fs;
wHat = 2*pi*f0/fs;
N0 = 30;
n = 0 : N0;
yofn = cos(wHat*n);
t = 0 : 0.01 : N0;
yoft = cos(wHat*t);
figure;
stem(n, yofn);
hold;
plot(t, yoft);
```



Epilogue: For a sinusoidal signal with discrete-time frequency $\widehat{\omega}_0 = 2\pi\frac{f_0}{f_s} = 2\pi\frac{N}{L}$ where the common factors in f_0 and f_s have been removed so that N and L are relatively prime, the discrete-time signal has a fundamental period of L samples. The fundamental period of L samples contains N periods of a continuous-time sinusoid with frequency f_0. Please see Handout D on Discrete-Time Periodicity.