

Solution Set for Homework #8

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November 23, 2021. [Version 3.0. Add frequency plots and selectivity for problem 8.3.](#)**PROBLEM 1: IDEAL DELAY.***Signal Processing First*, problem P-10.1, page 302.Derive the frequency response of an ideal delay system by substituting $x(t) = e^{j\omega t}$ into the system's definition

$$y(t) = x\left(t - \frac{1}{2}\right)$$

In addition, plot the magnitude of the frequency response for this system.

Solution: In this problem, we are able to observe the input and output signals for all time, i.e., $-\infty < t < \infty$. The output of the system is as follows:

$$y(t) = x\left(t - \frac{1}{2}\right) = e^{j\omega\left(t - \frac{1}{2}\right)} = e^{j\omega t} e^{-j\frac{\omega}{2}} = e^{-j\frac{\omega}{2}} e^{j\omega t} = H(j\omega) e^{j\omega t}$$

The frequency response of the system

$$H(j\omega) = e^{-j\frac{\omega}{2}}$$

is already in magnitude-phase form due to the implied 1 before the exponential term, i.e.

$$H(j\omega) = 1 e^{-j\frac{\omega}{2}} = |H(j\omega)| e^{j\angle H(j\omega)}$$

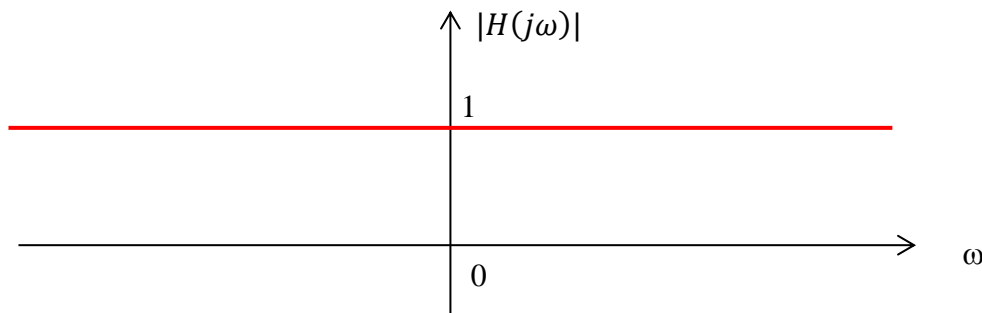
where

$$|H(j\omega)| = 1 \text{ and } \angle H(j\omega) = -\frac{\omega}{2}$$

Alternately, we can compute magnitude of the frequency response is calculated as follows:

$$H(j\omega) = e^{-j\frac{\omega}{2}} = \cos\left(\frac{\omega}{2}\right) - j\sin\left(\frac{\omega}{2}\right)$$

$$|H(j\omega)| = \sqrt{\left(\cos\left(\frac{\omega}{2}\right)\right)^2 + \left(\sin\left(\frac{\omega}{2}\right)\right)^2} = 1$$



Alternate Solution: We find the frequency response by taking the Fourier transform of the impulse response of the LTI system. Since we are able to observe the input and output signals for all time, i.e., $-\infty < t < \infty$, we do not have to worry about initial conditions.

The input response is obtained by inputting the Dirac delta, i.e. $x(t) = \delta(t)$, which gives output

$$h(t) = \delta\left(t - \frac{1}{2}\right)$$

The Fourier transform is

$$H(j\omega) = \int_{-\infty}^{\infty} \delta\left(t - \frac{1}{2}\right) e^{-j\omega t} dt = e^{-j\omega t} \Big|_{t=\frac{1}{2}} = e^{-j\frac{\omega}{2}}$$

which is the same frequency response as obtained in the original solution. The magnitude and phase responses are the same as above:

$$|H(j\omega)| = 1 \text{ and } \angle H(j\omega) = -\frac{\omega}{2}$$

Epilogue: The LTI ideal delay is a building block in continuous-time systems.

An LTI system with a constant non-zero magnitude response such as $|H(j\omega)| = 1$ passes all frequencies through equally well. This is called an *allpass filter*.

From the phase response, we can determine the group delay in seconds through the LTI system for a particular frequency by taking the derivative of the phase response and negating it. For a phase response of $\angle H(j\omega) = -\frac{\omega}{2}$, the group delay would be $\frac{1}{2}$ seconds, which is the delay in the ideal delay system. See also problem 8.2(b) below.

If we could only observe the ideal delay for time $t \geq 0$, then we would have to set the initial conditions to zero as a necessary condition for the ideal delay to be LTI. Please see [Handout U Property of Time-Invariance \(Shift-Invariance\) for a System Under Observation](#) for an example of an ideal delay under observation for time $t \geq 0$.

PROBLEM 2: CONTINUOUS TIME-FREQUENCY RESPONSE

Signal Processing First, problem P-10.9, page 305. In addition, for each of the seven filters given, describe the frequency selectivity in the magnitude response as lowpass, highpass, bandpass, bandstop, allpass, or notch.

Consider an LTI system whose frequency response $H(j\omega)$ is unknown. The system has a periodic input whose spectrum is shown in Fig. P-10.9.

For each part of this problem, the output of the system is given and the frequency response must be determined by selecting from the list numbered 1-7 below. Choose the frequency response $H(j\omega)$ of the system that could have produced the specific output when the input is the signal with the spectrum in Fig. P-10.9.

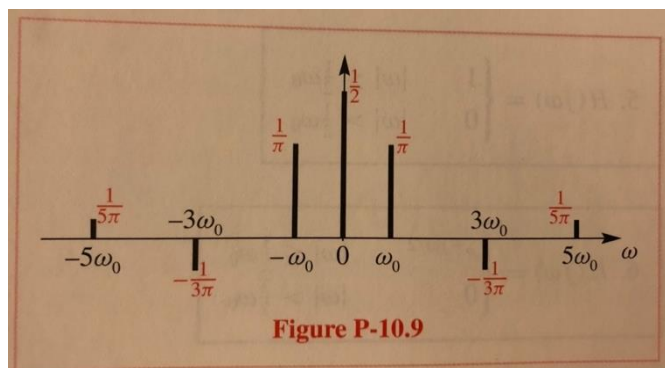


Figure P-10.9

Solution: Fig. P-10.9 plots the following Fourier series coefficients

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{1}{\pi}, & k = \pm 1 \\ 0, & k = \pm 2 \\ -\frac{1}{3\pi}, & k = \pm 3 \\ 0, & k = \pm 4 \\ \frac{1}{5\pi}, & k = \pm 5 \end{cases}$$

which can be used in the Fourier series formula

$$x(t) = \sum_{k=-5}^5 a_k e^{jk\omega_0 t} = \frac{1}{2} + \frac{2}{\pi} \cos(\omega_0 t) - \frac{2}{3\pi} \cos(3\omega_0 t) + \frac{2}{5\pi} \cos(5\omega_0 t)$$

Due to the system having linear and time-invariant properties, all the frequency components in the output signal had to be present in the input signal. That is, a linear time-invariant (LTI) system cannot create new frequencies.

Using LTI system properties, the output signal is simply the sum of the system's response to each frequency component of the input signal:

$$y(t) = \frac{1}{2} H(j0) + \frac{2}{\rho} \cos(\omega_0 t) H(j\omega_0) - \frac{2}{3\rho} \cos(3\omega_0 t) H(j3\omega_0) + \frac{2}{5\rho} \cos(5\omega_0 t) H(j5\omega_0)$$

We can write the frequency response into polar form as $H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)}$:

$$y(t) = \frac{1}{2} H(j0) + \frac{2}{\rho} |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0)) - \frac{2}{3\rho} |H(j3\omega_0)| \cos(3\omega_0 t + \angle H(j3\omega_0)) + \frac{2}{5\rho} |H(j5\omega_0)| \cos(5\omega_0 t + \angle H(j5\omega_0))$$

Please see lecture slide 14-6 and *Signal Processing First* Section 10-2.

a) $y(t) = \frac{1}{2}$

The output of system can be obtained by the following formula

$$y(t) = \sum_{k=-5}^5 H(jk\omega_0) a_k e^{jk\omega_0 t}$$

Hence, we can find the value of $H(j\omega)$ for frequencies that are present in the input.

$$H(jk\omega_0) = \begin{cases} 1, & k = 0 \\ 0, & k = \pm 1, \pm 3, \pm 5 \end{cases}$$

Filter 5 has similar response, so the input has passed through this lowpass filter.

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \frac{1}{2}\omega_0 \\ 0 & |\omega| > \frac{1}{2}\omega_0 \end{cases}$$

$$\text{b) } y(t) = \frac{1}{2} + \frac{2}{\pi} \cos\left[\omega_0\left(t - \frac{1}{2}\right)\right]$$

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0) a_k e^{jk\omega_0 t} = \frac{1}{2} + \frac{2}{\pi} \cos\left[\omega_0\left(t - \frac{1}{2}\right)\right]$$

$$H(jk\omega_0) = \begin{cases} 1, & k = 0 \\ e^{-j\omega_0/2}, & k = \pm 1 \\ 0, & k = \pm 3, \pm 5 \end{cases}$$

Filter 6 has this property and will give similar output. This lowpass filter removes frequencies above $\frac{3\omega_0}{2}$ and delays the input by $\frac{1}{2}$ sample. We can obtain the delay by computing the group delay for the filter as follows: $\text{Group Delay}(\omega) = -\frac{d}{d\omega} \angle H(j\omega) = -\frac{d}{d\omega} \left(-\frac{\omega}{2}\right) = \frac{1}{2}$.

$$H(j\omega) = \begin{cases} e^{-j\frac{\omega}{2}} & |\omega| \leq \frac{3}{2}\omega_0 \\ 0 & |\omega| > \frac{3}{2}\omega_0 \end{cases}$$

c)

$$y(t) = \frac{2}{\pi} \cos(\omega_0 t)$$

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0) a_k e^{jk\omega_0 t} = \frac{2}{\pi} \cos(\omega_0 t)$$

$$H(jk\omega_0) = \begin{cases} 1, & k = \pm 1 \\ 0, & k = 0, \pm 3, \pm 5 \end{cases}$$

The original signal has passed through a bandpass filter which removes all frequencies present in the input signal except ω_0 . Filter 7 shows a bandpass filter with this property.

$$H(j\omega) = \begin{cases} 1, & \frac{1}{2}\omega_0 < |\omega| < \frac{3}{2}\omega_0 \\ 0, & |\omega| < \frac{1}{2}\omega_0 \text{ or } |\omega| > \frac{3}{2}\omega_0 \end{cases}$$

d)

$$y(t) = x(t) - \frac{1}{2} = \frac{2}{\pi} \cos(\omega_0 t) - \frac{2}{3\pi} \cos(3\omega_0 t) + \frac{2}{5\pi} \cos(5\omega_0 t)$$

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0) a_k e^{jk\omega_0 t} = \frac{2}{\pi} \cos(\omega_0 t) - \frac{2}{3\pi} \cos(3\omega_0 t) + \frac{2}{5\pi} \cos(5\omega_0 t)$$

$$H(jk\omega_0) = \begin{cases} 0, & k = 0 \\ 1, & k = \pm 1, \pm 3, \pm 5 \end{cases}$$

This filter passes all frequencies except $\omega = 0$, therefore it acts as a highpass filter or a bandpass filter. Filter 1 is a highpass filter that can produce this output.

$$H(j\omega) = \begin{cases} 0 & |\omega| \leq \frac{1}{2}\omega_0 \\ 1 & |\omega| > \frac{1}{2}\omega_0 \end{cases}$$

e)

$$y(t) = x(t - \frac{1}{2})$$

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0) a_k e^{jk\omega_0 t}$$

$$H(jk\omega_0) = e^{-jk\omega_0/2}$$

Filter 2 gives this response, which is allpass.

$$H(j\omega) = e^{-j\omega/2}$$

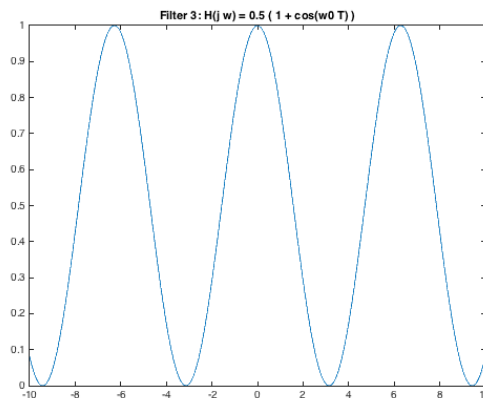
Filters 3 and 4 cannot produce any of output signals.

Filter 3:

$$H(j\omega) = \frac{1}{2} [1 + \cos(\omega T_0)]$$

$$\text{for } \omega = k\omega_0 \rightarrow H(jk\omega_0) = \frac{1}{2} [1 + \cos(k\omega_0 T_0)] = \frac{1}{2} [1 + \cos(2\pi k)] = 1$$

For $\omega = k\omega_0$, this filter passes all the harmonic frequencies; however, it rejects sub-harmonic frequencies at $\omega = \frac{k\omega_0}{2}$. This filter has a periodic magnitude response as shown above.



Filter 4: This filter removes frequencies above $\frac{3\omega_0}{2}$ and is a lowpass filter.

PROBLEM 3: FORWARD CONTINUOUS TIME FOURIER TRANSFORM

Compute the continuous-time Fourier transform $X(j\omega)$ for continuous-time signal $x(t)$ using the definition in Signal Processing First in equation (11.1)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

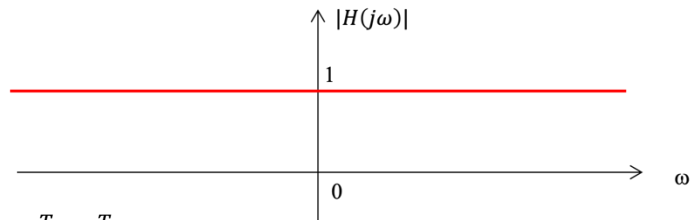
for the following time-domain signals $x(t)$. In addition, for each part, describe the frequency selectivity of the magnitude response as lowpass, highpass, bandpass, bandstop, allpass, or notch.

Solution:

a) $x(t) = \delta(t)$. We can use the sifting property for the Dirac delta:

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t} dt = e^{-j\omega t} \Big|_{t=0} = 1$$

Allpass filter. All frequencies pass through equally well. See Problem 8.1. Magnitude response plotted on the right.



b) Rectangular pulse of unit amplitude that lasts from $-\frac{T}{2}$ to $\frac{T}{2}$ seconds.

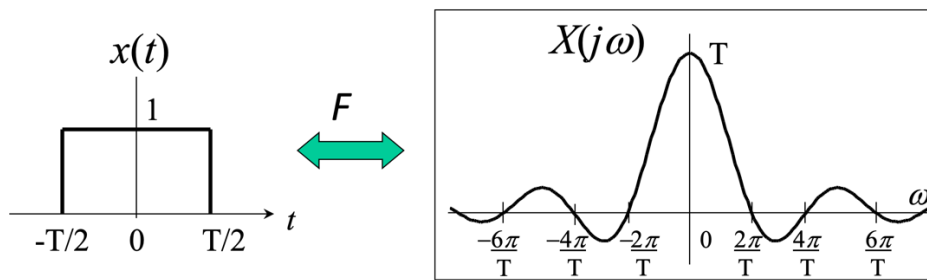
$$x(t) = u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \left[u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right) \right] e^{-j\omega t} dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\omega t} dt$$

$$X(j\omega) = -\frac{1}{j\omega} \left[e^{-j\omega t} \right]_{-\frac{T}{2}}^{\frac{T}{2}} = -\frac{1}{j\omega} \left(e^{-j\omega \frac{T}{2}} - e^{j\omega \frac{T}{2}} \right) = \frac{2j \sin\left(\frac{T}{2}\omega\right)}{j\omega} = T \frac{\sin\left(\frac{T}{2}\omega\right)}{\frac{T}{2}\omega}$$

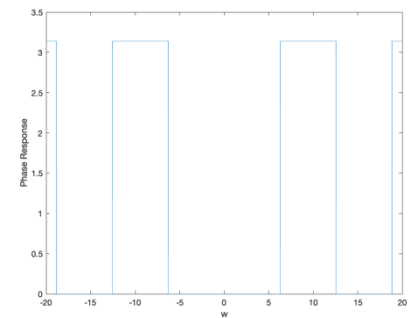
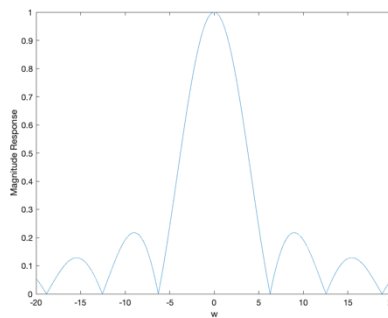
$$X(j\omega) = T \operatorname{sinc}\left(\frac{\omega T}{2\pi}\right) \text{ where } \operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

See also *Signal Processing First*, pp. 314-315. **Lowpass filter.** Lecture slide 15-6:



Magnitude and phase plots in MATLAB for $T = 1$ for $-20 < \omega < 20$ rad/s. In the magnitude response, the first zero to the right of zero frequency occurs at $\omega = \frac{2\pi}{T} = 2\pi$. The phase response is π whenever the sinc pulse goes negative.

```
T = 1;
w = -20 : 0.01 : 20;
X = T*sinc(w*T/(2*pi));
figure;
plot(w, abs(X));
xlabel('w');
ylabel('Magnitude Response');
figure;
plot(w, angle(X));
xlabel('w');
ylabel('Phase Response');
```

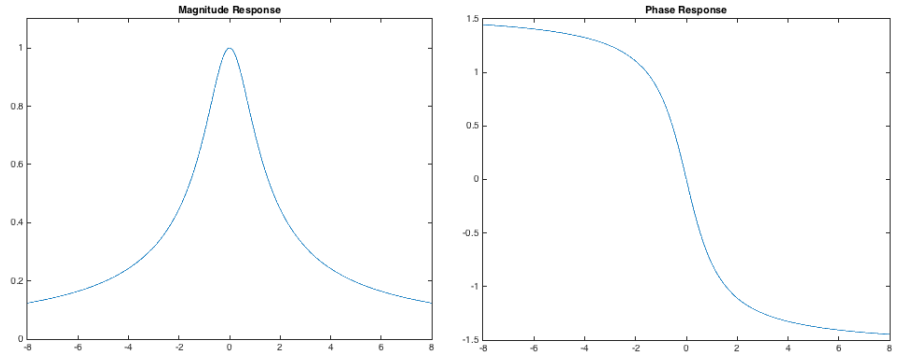


c) $x(t) = e^{-at}u(t)$ for positive and real-valued a

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt = -\frac{1}{a+j\omega} [e^{-(a+j\omega)t}]_0^{\infty} = \frac{1}{a+j\omega}$$

See also *Signal Processing First*, pp. 309 & 313. **Lowpass filter.** From Lecture Slide 15-5 for $a = 1$:

```
w = -8 : 0.01 : 8;
H = 1 ./ (1 + j*w);
Hmag = abs(H);
Hphase = phase(H);
figure;
plot(w, Hmag);
title('Magnitude Response');
ylim( [-0.0 1.1] );
figure;
plot(w, Hphase);
title('Phase Response');
```

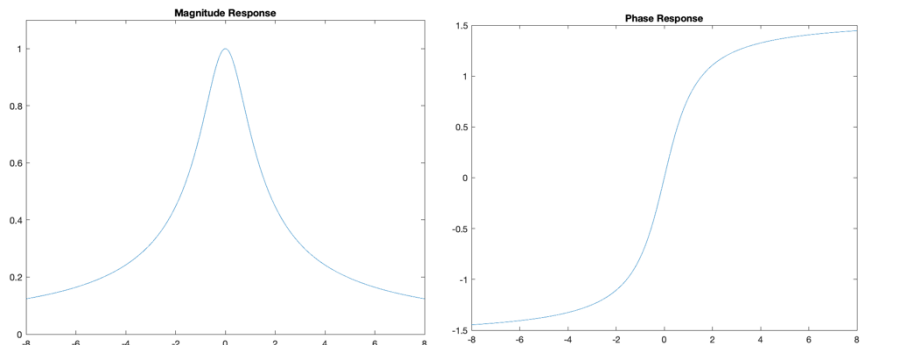


d) $x(t) = e^{bt}u(-t)$ for positive and real-valued b

$$X(j\omega) = \int_{-\infty}^{\infty} e^{bt}u(-t)e^{-j\omega t} dt = \int_{-\infty}^0 e^{(b-j\omega)t} dt = \frac{1}{b-j\omega} [e^{(b-j\omega)t}]_{-\infty}^0 = \frac{1}{b-j\omega}$$

See also *Signal Processing First*, pp. 314. **Lowpass filter.** Matlab plots for the magnitude and phase for $b = 1$. Note that the magnitude response is the same as part (c) and the phase response is flipped in frequency:

```
w = -8 : 0.01 : 8;
H = 1 ./ (1 - j*w);
Hmag = abs(H);
Hphase = phase(H);
figure;
plot(w, Hmag);
title('Magnitude Response');
ylim( [-0.0 1.1] );
figure;
plot(w, Hphase);
title('Phase Response');
```



e) $x(t) = e^{-a|t|}$ for $-\infty < t < \infty$ for positive and real-valued a . When $t < 0$, $|t| = -t$. When $t > 0$, $|t| = t$. We can reuse the results from parts (c) and (d).

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|}e^{-j\omega t} dt = \int_{-\infty}^0 e^{-a(-t)}e^{-j\omega t} dt + \int_0^{\infty} e^{-at}e^{-j\omega t} dt$$

$$X(j\omega) = \int_{-\infty}^0 e^{at}e^{-j\omega t} dt + \int_0^{\infty} e^{-at}e^{-j\omega t} dt$$

$$X(j\omega) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a-j\omega+a+j\omega}{(a+j\omega)(a-j\omega)} = \frac{2a}{a^2+\omega^2}$$

Lowpass filter. See the magnitude response plot on the right for $a = 1$:

