

Solution Set for Homework #2 Version 1.0

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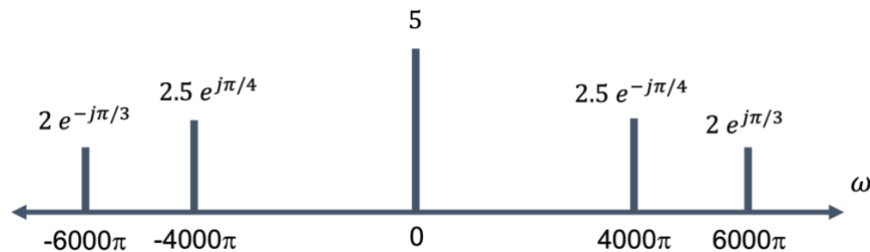
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Takeaways: Fourier analysis reveals what frequency components are in a time-domain signal and how strong they are, and Fourier synthesis converts the frequency components into a time-domain signal. The representation of the frequency components has a natural visual representation, which can help guide the interpretation and analysis of the signal. Within Fourier analysis, Fourier series assumes the time-domain signal is periodic. Fourier series analysis has a key drawback – it does not tell us *when* the frequency components occur in time. We can find when the frequency components occur in time by using time-frequency analysis algorithms such as spectrograms.

PROBLEM 1

Prologue: This problem asks to generate the signal in the time domain from the signal's frequency domain representation (spectrum). The spectral and time representations give complementary views into the signal.

Problem: The spectrum is



Solution: The above spectrum indicates that the signal has frequency components of -6000π rad/s, -4000π rad/s, 0 rad/s, 4000π rad/s and 6000π rad/s. The strongest frequency component is at 0 rad/s because its magnitude (5) is the largest. The second strongest frequency components are at -4000π rad/s and 4000π rad/s. The weakest frequency components are at -6000π rad/s and 6000π rad/s.

Part (a) We can read off the spectral components to create the time-domain representation of the signal

$$\begin{aligned} x(t) &= 2e^{-j\frac{\pi}{3}}e^{-j2\pi(3000)t} + 2.5e^{j\frac{\pi}{4}}e^{-j2\pi(2000)t} + 5e^{j0t} + 2.5e^{-j\frac{\pi}{4}}e^{j2\pi(2000)t} + 2e^{j\frac{\pi}{3}}e^{j2\pi(3000)t} \\ &= 5 + \frac{5e^{j\frac{\pi}{4}}e^{-j2\pi(2000)t} + 2.5e^{-j\frac{\pi}{4}}e^{j2\pi(2000)t}}{2} + \frac{4e^{-j\frac{\pi}{3}}e^{-j2\pi(3000)t} + 4e^{j\frac{\pi}{3}}e^{j2\pi(3000)t}}{2} \\ &= 5 + 5 \cos\left(2\pi(2000)t - \frac{\pi}{4}\right) + 4 \cos\left(2\pi(3000)t + \frac{\pi}{3}\right) \end{aligned}$$

Part (b) The signal $x(t)$ is periodic with period T , if $x(t + T) = x(t)$ for all values of t :

$$\begin{aligned} x(t) &= 5 + 5 \cos\left(2\pi(2000)t - \frac{\pi}{4}\right) + 4 \cos\left(2\pi(3000)t + \frac{\pi}{3}\right) \\ x(t + T) &= 5 + 5 \cos\left(4000\pi t + 4000\pi T - \frac{\pi}{4}\right) + 4 \cos\left(6000\pi t + 6000\pi T + \frac{\pi}{3}\right) \end{aligned}$$

In order for $x(t + T) = x(t)$ for all values of t , we can equate the first terms in $x(t + T)$ and $x(t)$, equate the second terms in $x(t + T)$ and $x(t)$, and equate the third terms in $x(t + T)$ and $x(t)$, and find values of T that work for all three terms.

The first terms are already equal to each other (5). The second terms are equal when $2\pi(2000)T = 2\pi k$ where k is an integer; i.e., when T is $1/2000$, **2/2000**, $3/2000$, etc. The third terms are equal when $2\pi(3000)T = 2\pi m$ where m is an integer, i.e. when T is $1/3000$, $2/3000$, **3/3000**, $4/3000$, $5/3000$, $6/3000$, $7/3000$, $8/3000$, etc. The common values for

the period T in seconds among all three terms are $1/1000$, $2/1000$, $3/1000$, $4/1000$, etc., which means that the fundamental period is 1 ms, and the fundamental frequency is 1000 Hz.

Alternately, the fundamental frequency could be found by computing the greatest common divisor (gcd) of 2000 Hz and 3000 Hz: $f_0 = \text{gcd}(2000 \text{ Hz}, 3000 \text{ Hz}) = 1000 \text{ Hz}$ and $T_0 = 1/f_0 = 1/(1000 \text{ Hz}) = 1 \text{ ms}$.

Part (c) $\cos(\Theta) = \text{Re}\{e^{j\Theta}\}$. From the inverse Euler's formula,

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Therefore, sinusoids should be shown by both frequencies.

PROBLEM 2

Prologue: This problem concerns amplitude modulation which is the basis for AM/FM radio, Wi-Fi, cellular, cable modems, and other communication systems. Amplitude modulation can also be used to create audio effects.

Modulation Property. From this problem, you'll be able to derive a fundamental principle called the Modulation Property. It says that multiplying a signal $v(t)$ by $\cos(2\pi f_c t)$ in the time domain will in the frequency domain shift the frequency content in $v(t)$ by f_c and by $-f_c$, as shown in the solution. Modulation allows the conversion of an audio signal $v(t)$ into a wireless electromagnetic radio frequency (RF) signal centered at f_c for long-range broadcast across the state and country, or a communication signal into the 2.4 GHz RF band for Wi-Fi. Example carrier frequencies: AM radio ~1 MHz, FM radio ~100 MHz, and Wi-Fi 2.4-2.499 GHz. (There are other Wi-Fi bands.) Audio signals have content between 20 Hz and 20 kHz, which are low frequencies compared to RF carrier frequencies. Audio signals don't propagate very far; e.g. sound from outdoor concert venues might propagate a few kilometers. By converting the audio signal to an AM radio signal, we can broadcast it for thousands of kilometers.

Problem: In AM radio, the transmitted signal (voice or music) is modulated by a sinusoid at the assigned broadcast frequency of the AM station. For example, the Zone Austin Sports Talk in Austin has a radio station frequency (carrier frequency) f_c of 1300 kHz. In the US, the AM radio band is 540 kHz to 1700 kHz.

If $x(t)$ is the voice/music signal, then the transmitted signal would be:

$$y(t) = (x(t) + A) \cos(2\pi f_c t)$$

where A is a constant. When A is chosen so that $x(t) + A \geq 0$, the receiver design can be simplified to a squaring block (single transistor) plus accompanying circuitry.

Suppose the signal to be transmitted $x(t)$ is an 'E' in the fifth octave on the Western scale:

$$x(t) = 2 \cos(2\pi(660)t + \pi/2)$$

Draw the spectrum for $y(t)$ assuming a carrier frequency of 1300 kHz with $A = 2$. Hint: Substitute for $x(t)$ and expand $y(t)$ into a sum of cosine terms of three different frequencies.

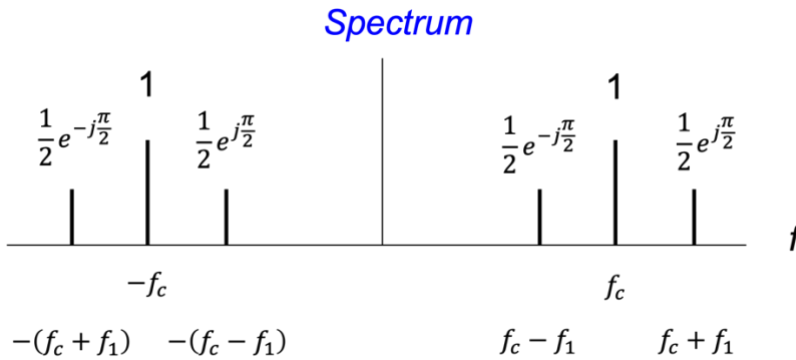
Solution: Let $f_1 = 660 \text{ Hz}$, $f_1 = 1300 \text{ kHz}$, and $A = 2$,

$$y(t) = \left(2 \cos\left(2\pi f_1 t + \frac{\pi}{2}\right) + A\right) \cos(2\pi f_c t) = 2 \cos\left(2\pi f_1 t + \frac{\pi}{2}\right) \cos(2\pi f_c t) + A \cos(2\pi f_c t)$$

$$\text{First term} = 2 \left[\frac{1}{2} \left(e^{j\frac{\pi}{2}} e^{j2\pi f_1 t} + e^{-j\frac{\pi}{2}} e^{-j2\pi f_1 t} \right) \right] \left[\frac{1}{2} \left(e^{j2\pi f_c t} + e^{-j2\pi f_c t} \right) \right]$$

$$\text{First term} = \frac{1}{2} \left(e^{j\frac{\pi}{2}} e^{j2\pi(f_c+f_1)t} + e^{-j\frac{\pi}{2}} e^{j2\pi(f_c-f_1)t} + e^{j\frac{\pi}{2}} e^{-j2\pi(f_c-f_1)t} + e^{-j\frac{\pi}{2}} e^{-j2\pi(f_c+f_1)t} \right)$$

$$\text{Second Term} = \frac{A}{2} e^{j2\pi f_c t} + \frac{A}{2} e^{-j2\pi f_c t} = e^{j2\pi f_c t} + e^{-j2\pi f_c t}$$



Spectrum is not drawn to scale.

PROBLEM 3

Prologue: Chirp Signals. In a chirp signal, as mentioned in Section 3-8 of *Signal Processing First*, the principal frequency increases (or decreases) with time.

Active sonar systems. The transmitter sends out a “ping” as sound in form of a chirp and then receives sound. The time elapsed between the transmission and reception of the chirp indicates the roundtrip time experienced by the signal after bouncing off an object in the water and returning to the receiver. By receiving sounds in different directions using multiple microphones, the sonar can build a map of the objects in the water.

Bats use chirps for echolocation. Pipestrelle bats uses [chirps that sweep down from 70 to 45 kHz](#).

Test & Measurement. When one measures the response of a system to different frequencies, a time-consuming approach is to input a single sinusoid, measure the output, and repeat using many different frequencies. Instead, inputting a chirp can allow the measurement to be performed in one take.

4G/5G cellular communication systems periodically send a Zadoff-Chu chirp sequence to help measure the distortion in the electromagnetic propagation from transmitter to receiver.

Prologue: Spectrograms. For a periodic signal, Fourier series analysis tells us what frequencies are present and in what strengths. However, the analysis cannot tell us when in time a particular frequency occurs and at what strength.

Consider the Fourier series analysis of a square wave on lecture slide 3-11 with a fundamental period of 40ms. Fourier series analysis says that the square wave contains a frequency at 25 Hz with a strength of $-j/\pi$. However, the square wave has an amplitude of 0 from 20ms to 40ms, and hence, no frequency components are present in this time interval. The Fourier series analysis computes the *average* strength of a frequency component over the fundamental period.

A spectrogram can be used to analyze both periodic and non-periodic signals in the time and frequency domains simultaneously. The spectrogram indicates when in time a particular frequency occurs and at what strength.

Problem: This problem analyzes the chirp signal, which is used in sonar and radar systems, IEEE 802.11az Wi-Fi indoor positioning, test and measurement, and 4G/5G cellular communications. The chirp signal is a sinusoid whose principal frequency increases (or decreases) over time. A chirp signal has the form

$$c(t) = \cos(\theta(t)) \text{ where } \theta(t) = 2\pi (f_0 + \frac{1}{2}f_{\text{step}} t) t = 2\pi f_0 t + \pi f_{\text{step}} t^2$$

The principal frequency is f_0 when $t = 0$ and then changes over time at a rate of f_{step} in units of Hz/s. The principal frequency of a sinusoid at a given point in time is called the *instantaneous frequency*, and it is defined as

$$\frac{d\theta(t)}{dt} = \frac{\text{Change in angle (radians)}}{\text{Change in time (seconds)}} = 2\pi f_0 + 2\pi f_{\text{step}} t = 2\pi (f_0 + f_{\text{step}} t)$$

and has units of rad/s which is a frequency measure. We can divide by 2π to convert from rad/s to Hz.

Part (a): Generate a chirp signal that lasts 10s with $f_0 = 220$ Hz and $f_{\text{step}} = 400$ Hz/s. Use sampling rate f_s of 44100 Hz. The chirp will sweep from ‘A’ in the third octave (220 Hz) to the highest note (4186 Hz) on an 88-key piano

https://en.wikipedia.org/wiki/Piano_key_frequencies. We are starting at 220 Hz because many playback systems on laptops do not play frequencies below 220 Hz.

```
time = 10;      % length of time in seconds
f0 = 220;      % specify starting principal frequency
fstep = 400;   % specify frequency slope
fs = 44100;    % sampling rate
Ts = 1/fs;     % sampling time: time interval between samples
t = 0 : Ts : time; % create a time vector
theta = 2*pi*(f0+0.5*fstep*t).*t;
y = cos(theta); % create chirp waveform
```

Part (b): The chirp signal can be played as an audio signal in MATLAB:

```
sound(y, fs); % play back chirp signal
```

The chirp linearly sweeps frequencies from 220 to 4220 Hz, and sounds like a note increasing in “pitch” over time. The sweep spans [five octaves of A notes on a Western scale](#): 220, 440, 880, 1760, and 3520 Hz.

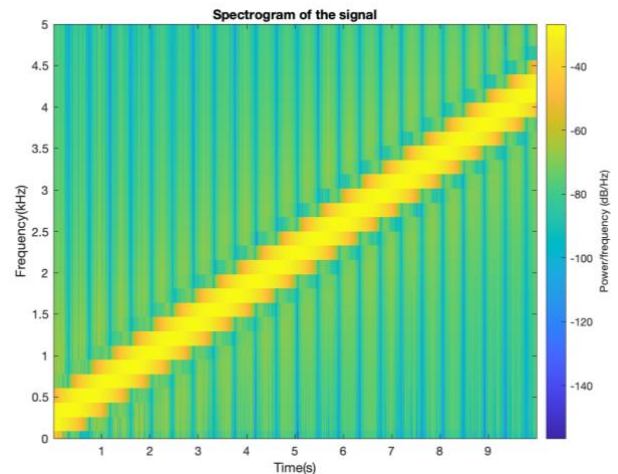
Part (c): We can visualize the variation of the principal frequency over time by using a time-frequency representation. The time-frequency representation below is called a spectrogram.

The spectrogram for the chirp signal (next page) shows the principal frequency changing over time with a linear slope. The principal frequency is initially at 220 Hz and increases linearly to 4220 Hz. The principal frequency has the highest magnitude at every instant of time throughout the entire duration of the chirp signal, which is shown in yellow.

As will be described on lecture slides 4-8 and 4-9, a spectrogram takes the first N_{win} samples of the signal, weights the values (using a rectangular pulse by default), computes the Fourier series coefficients, and plots the magnitude of the Fourier series coefficients vertically. The spectrogram then shifts the time signal to the right and repeats the previous steps using a block of the next N_{win} samples. The frequency resolution of the spectrogram is f_s / N_{win} .

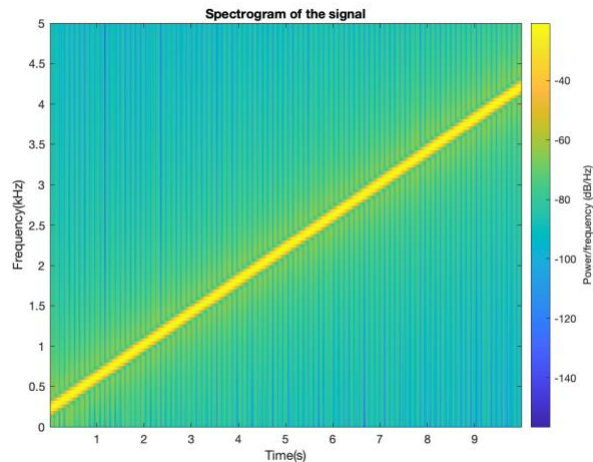
In the code below, a Hamming window is used to weight the values in each block of samples, and we zoom in on the frequency range of 0 to 5 kHz in spectrogram using the `yylim` command. By default, the spectrogram only shows non-negative frequencies:

```
figure;
spectrogram(y, hamming(256), 128, 256, fs, 'yaxis')
ylim([0,5]);
title('Spectrogram of the signal');
ylabel('Frequency (kHz)');
xlabel('Time (s)');
```



Although not asked, we can improve the frequency resolution (i.e., the accuracy of frequency values observed in the spectrogram) by increasing the block size:

```
figure;
spectrogram(y, hamming(1024), 512, 1024, fs, 'yaxis')
ylim([0,5]);
title('Spectrogram of the signal');
ylabel('Frequency (kHz)');
xlabel('Time (s)');
```



PROBLEM 4

Prologue: This problem explores various audio effects by applying nonlinear operations to a sinusoidal signal. It also explores what happens to the frequency component of a sinusoidal signal due to nonlinear operations. This is a common approach in analyzing how a nonlinear system (e.g. a diode or transistor) responds to different frequencies. Among the nonlinear operations, part (a) can be used in amplitude demodulation (see the Prologue for Problem 2).

Problem: Consider the signal $x(t) = \cos(2\pi f_0 t)$ where $f_0 = 440$ Hz is an 'A' in the 4th octave on the Western scale.

Write MATLAB code to implement the following audio effects. Play each audio signal for 5 seconds over $0 \leq t \leq 5$ and use a sampling rate of 8000 Hz:

- $y(t) = x^2(t)$. Used in power calculations, image display, communication systems.
- $y(t) = \exp(x(t))$. Used as a penalty function in machine learning training algorithms.

For each part,

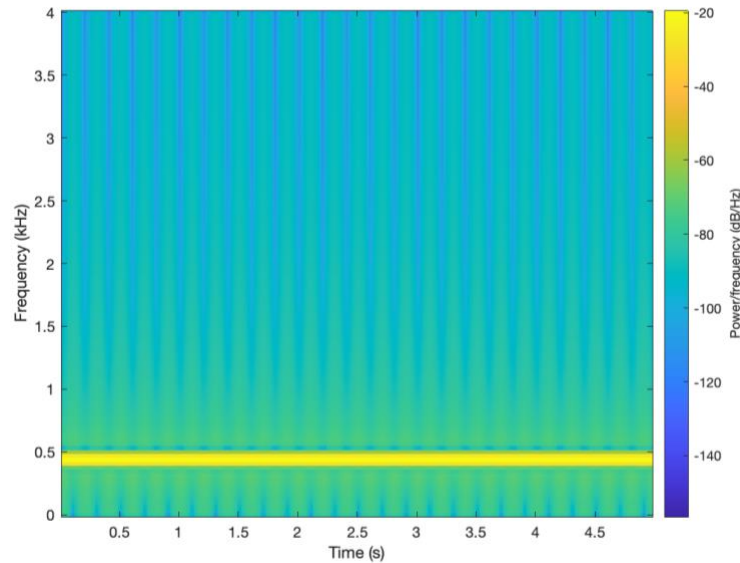
- give a mathematical analysis to determine what frequencies are present in $y(t)$
- plot the spectrogram and describe the principal frequency components and how they compare to your analysis in part (i)
- play $y(t)$ as an audio signal by subtracting the average value (because we cannot hear DC) and then using the `soundsc` command on the remaining signal, and describe what you hear vs. $x(t)$.

Please submit the MATLAB code that you have written.

Solution:

MATLAB code for generating, plotting the time-frequency representation, and playing as sound $x(t)$:

```
f0 = 440;
fs = 8000;
Ts = 1/fs;
t = 0:Ts:5;
x = cos(2*pi*f0*t);
figure
spectrogram(x, hamming(256), 128, 256, fs, 'yaxis')
sound(x, fs)
pause(8);
```



$x(t)$ only contains frequencies ± 440 Hz, and the spectrogram shows this. The spectrogram, by default, only shows non-negative frequencies, and the plot over negative frequencies will be a mirror image of the positive frequencies.

Note: We're sampling a continuous-time signal at a sampling rate of $f_s = 8000$ Hz. From the sampling theorem, $f_s > 2 f_{\max}$ or by dividing each side by 2, $f_{\max} < (1/2) f_s$. By sampling, we can only capture continuous-time frequencies up to $(1/2) f_s$, or 4000 Hz. We won't be able to see harmonics at or above 4000 Hz in the spectrogram.

Part (a):

$$y(t) = \cos^2(2\pi 440t)$$

$$y(t) = \frac{1}{2} + \frac{1}{2} \cos(2\pi 880t)$$

Using the trigonometric identity $\cos(2\theta) = 2\cos^2(\theta) - 1$

$y(t)$ contains 0 Hz and ± 880 Hz, which is DC (zero frequency) plus the second harmonic of 440 Hz.

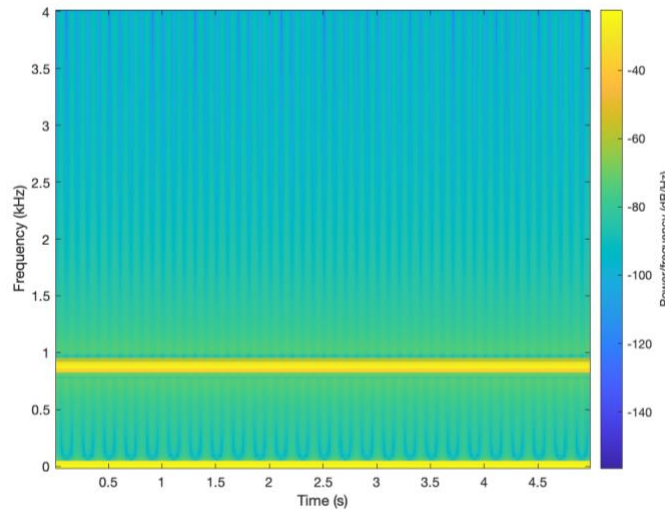
Alternatively,

$$y(t) = \cos^2(2\pi 440t)$$

$$y(t) = \frac{(e^{j2\pi 440t} + e^{-j2\pi 440t})^2}{4} = \frac{e^{j2\pi 880t} + e^{-j2\pi 880t} + 2}{4} = \frac{1}{2} + \frac{\cos(2\pi 880t)}{2}$$

using the binomial expansion for $(a+b)^2 = a^2 + b^2 + 2ab$.

```
f0 = 440;
fs = 8000;
Ts = 1/fs;
t = 0:Ts:5;
x = cos(2*pi*f0*t);
ya = x.^ 2;
figure(1)
spectrogram(ya, hamming(256), 128, 256, fs, 'yaxis')
ya_no_mean = ya - mean(ya);
soundsc(ya_no_mean, fs);
pause(8);
```



It sounds ‘thin’, which is because just one frequency is audible in the waveform, while in part (b) below, many harmonics could be heard.

Part (b): Using Taylor series expansion about $x = 0$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{so} \quad y(t) = e^{x(t)} = \sum_{n=0}^{\infty} \frac{x^n(t)}{n!} = 1 + x(t) + \frac{x^2(t)}{2!} + \frac{x^3(t)}{3!} + \frac{x^4(t)}{4!} + \dots$$

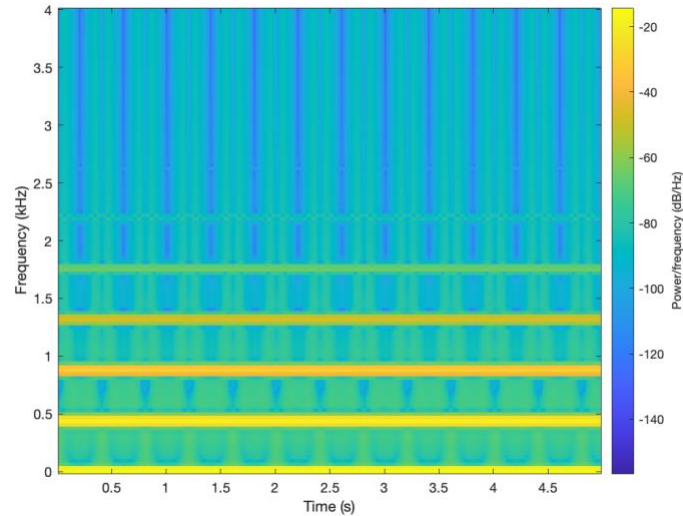
As shown in the above formula, $y(t)$ includes $x^2(t)$. When $x(t) = \cos(2\pi 440t)$, as analyzed in part (a) of this problem, $x^2(t)$ contains frequencies of -880 Hz, 0, and 880 Hz with strengths of $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$, respectively. The frequencies correspond to the zeroth and second harmonic of 440 Hz.

In addition, $y(t)$ includes $x^3(t)$. Lecture slide 3-9 shows that when $x(t) = \cos(2\pi 440t)$, the frequency components of $x^3(t)$ are -1320 Hz, -440, 440 Hz, and 1320 Hz with strengths of $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$ and $\frac{1}{8}$, respectively. The frequencies correspond to the first and third harmonic of 440 Hz.

Also, $y(t)$ includes $x^4(t)$. When $x(t) = \cos(2\pi 440t)$, as analyzed below, $x^4(t)$ contains frequencies of -1760 Hz, -880 Hz, 0, 880 Hz, and 1760 Hz with strengths of $\frac{1}{16}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{4}$, and $\frac{1}{16}$ respectively. The frequencies correspond to the zeroth, second, and fourth harmonic of 440 Hz.

So, when $x(t) = \cos(2\pi f_0 t)$, the signal $x(t) = \cos^n(2\pi f_0 t)$ will have odd harmonics up to the n th harmonic if n is odd, and even harmonics up to the n th harmonic if n is even including a zero-frequency component.

```
f0 = 440;
fs = 8000;
Ts = 1/fs;
t = 0:Ts:5;
x = cos(2*pi*f0*t);
yb = exp(x);
figure(1)
spectrogram(yb, hamming(256), 128, 256, fs, 'yaxis')
yb_no_mean = yb - mean(yb);
soundsc(yb_no_mean, fs);
pause(8);
```



I hear multiple frequencies. A couple of the frequencies sound like beat frequencies.

Fourth-order nonlinearity. Although not asked, here's the analysis for a fourth-order nonlinearity:

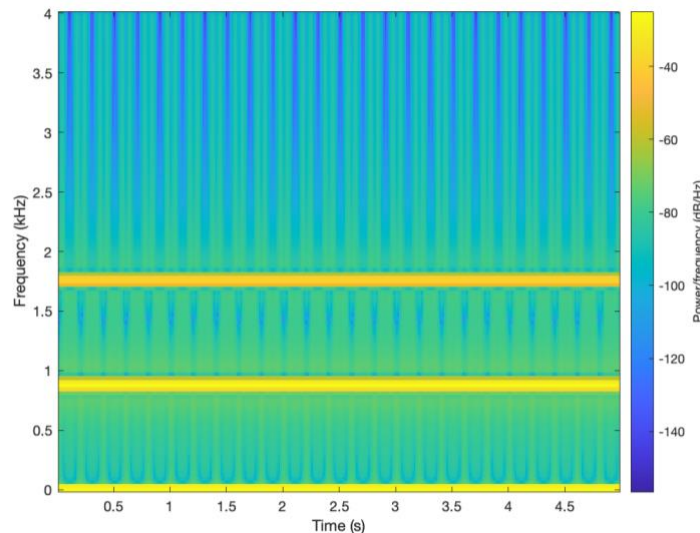
$$y(t) = x^4(t) = \cos^4(2\rho f_0 t) = \left(\frac{e^{j2\rho f_0 t} + e^{-j2\rho f_0 t}}{2} \right)^4 = \frac{e^{j8\rho f_0 t} + e^{-j8\rho f_0 t} + 4e^{j4\rho f_0 t} + 4e^{-j4\rho f_0 t} + 6}{16}$$

$$= 0.375 + 0.5\cos(4\rho f_0 t) + 0.125\cos(8\rho f_0 t) = 0.375 + 0.5\cos(2\rho(880)t) + 0.125\cos(2\rho(1760)t)$$

By using the binomial expansion for $(a + b)^4 = a^4 + b^4 + 4 a^3 b + 4 a b^3 + 6a^2b^2$.

$y(t)$ contains frequencies 0 Hz, ± 880 Hz and ± 1760 Hz, i.e., DC plus the second and fourth harmonics of 440 Hz.

We hear 880 Hz ('A' in the fifth octave) and 1760Hz ('A' in the sixth octave).



Epilogue: Harmonics of a fundamental frequency give the note at the fundamental frequency a richer sound. When playing a note, an instrument creates harmonics, and the harmonics can help one identify which instrument it is. When $x(t) = \cos(2\pi f_0 t)$, the signal $x(t) = \cos^n(2\pi f_0 t)$ will have odd harmonics up to the n th harmonic if n is odd, and even harmonics up to the n th harmonic if n is even including a zero-frequency component.

References

James H. McClellan, Ronald W. Schafer & Mark A. Yoder, *Signal Processing First*, Prentice-Hall, ISBN 978-0130909992, 2003. [Errata](#). [On-line Companion](#).

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