Lecture 5 Sampling & Aliasing Part 3
Handout D Discrete-Time Periodicity
Handout N Derivation of Fourier series

The University of Texas at Austin
EE 313 Linear Systems and Signals
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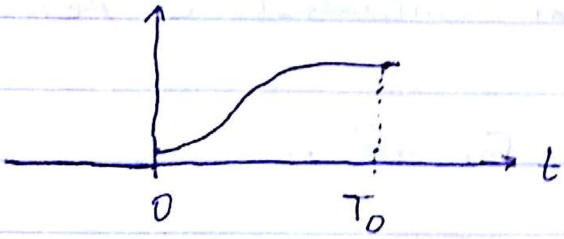
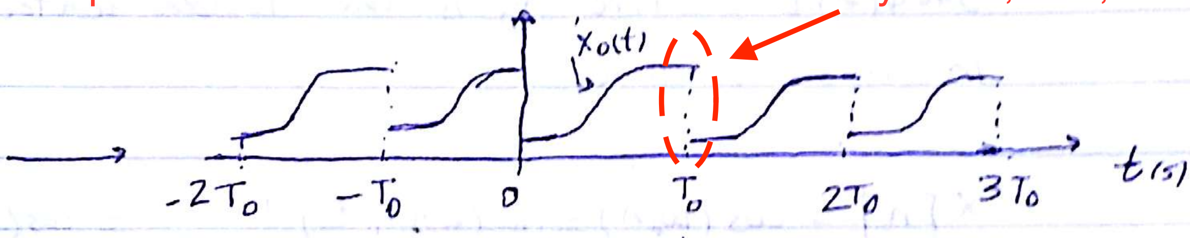
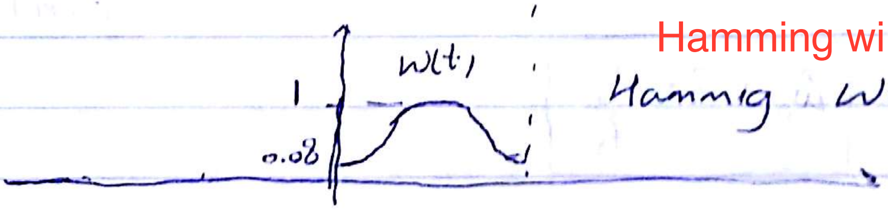


x(t) periodically extended

Periodic replication introduces an artificial discontinuity at T0, 2T0, etc.





Hamming window
Hammig window

Hamming window

In Hammig window, middle values gets the highest weight and values at edges recieve the least weight

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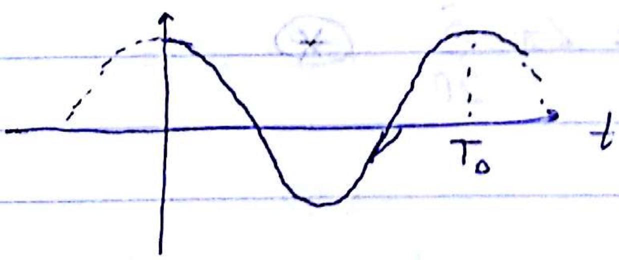
Continous Time Periodicity



$x(t+T) = x(t)$ for all t

$x(t) = \cos(2\pi f t) \Rightarrow$ fundamental period is $T_0 = \frac{1}{f_0}$

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Discrete - Time Periodicity

It's different from continuos - time periodicity.

$x[n+N_0] = x[n]$ for all n

smallest positive N_0 is the fundamental period in discrete-time

$x[n] = \cos(\hat{\omega}_0 n) = \cos(2\pi f_0 t)\Big|_{t=nT_s} = \cos(\underbrace{2\pi \frac{f_0}{f_s}}_{\hat{\omega}_0} n)$

$$\boxed{\hat{\omega}_0 = 2\pi \frac{f_0}{f_s}}$$

$\hat{\omega}_0 = 2\pi \frac{f_0}{f_s} = 2\pi \frac{N}{L}$, N, L are relatively prime. integers.

Example: $f_0 = 1200$ (Hz), $f_s = 8000$ (Hz)

$\hat{\omega}_0 = 2\pi \frac{1200}{8000} = 2\pi \frac{3}{20}$ (*)

$x[n] = \cos(\hat{\omega}_0 n)$

$x[n+N_0] = x[n]$ for all n

$x[n+N_0] = \cos(\hat{\omega}_0 (n+N_0)) = \cos(2\pi \frac{N}{L}(n+N_0))$

$= \cos(2\pi \frac{N}{L} n + 2\pi \frac{N}{L} N_0)$

$$2\pi \frac{N}{L} N_0 = 2\pi k$$

$$N_0 = L \Rightarrow 2\pi N = 2\pi k \Rightarrow k = N$$

$$\Rightarrow x[n+N_0] = \cos\left(2\pi \frac{N}{L} n + 2\pi k\right) = \cos\left(2\pi \frac{N}{L} n\right)$$

Sample at sampling rate f_s and $f_s > 2f_0$

f_0 $\quad \hat{\omega}_0$

$x(t) = \cos(2\pi f_0 t)$

sampling captures $\quad f_0 < \frac{1}{2} f_s$

$$-\frac{1}{2} f_s \leq f_0 \leq \frac{1}{2} f_s$$

$$\hat{\omega}_0 = 2\pi \frac{f_0}{f_s}$$

f_0	$\hat{\omega}_0$
$\frac{1}{2} f_s$	π
$-\frac{1}{2} f_s$	$-\pi$

f_s $\quad 2\pi$

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continue from Example

$f_0 = 1200$, $f_s = 8000 \rightarrow \hat{\omega}_0 = 2\pi \frac{3}{20}$

$f_0 = 2000$, $f_s = 8000 \rightarrow \hat{\omega}_0 = 2\pi \frac{1}{4}$

please see Handout D on Discrete-Time Periodicity for more info

Derivation of the Fourier series synthesis formula

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt}$$

Multiply both side by $e^{-j\frac{2\pi}{T_0}\ell t}$ and integrate

$$\int_0^{T_0} x(t)\, e^{-j\frac{2\pi}{T_0}\ell t}\, dt = \int_0^{T_0} \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt} e^{-j\frac{2\pi}{T_0}\ell t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^{T_0} e^{j\frac{2\pi}{T_0}(k-\ell)t}\, dt$$

Two cases:

$k=\ell$: $\displaystyle\int_0^{T_0} e^{j0t}\, dt = \int_0^{T_0} 1\, dt = T_0$

$k\neq\ell$: $m = k-\ell \longrightarrow \displaystyle\int_0^{T_0} e^{j\frac{2\pi}{T_0}mt}\, dt = \left.\frac{e^{j\frac{2\pi}{T_0}mt}}{j\frac{2\pi}{T_0}m}\right|_0^{T_0}$

$$= \frac{e^{j\frac{2\pi}{T_0}mT_0} - 1}{j\frac{2\pi}{T_0}m} = 0$$

$$e^{j2\pi m} = \cos(2\pi m) + j\sin(2\pi m) = 1 + j0 = 1$$

$$\int_0^{T_0} x(t)\, e^{-j\frac{2\pi}{T_0}\ell t}\, dt = a_\ell T_0$$

$$a_\ell = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j\frac{2\pi}{T_0}\ell t}\, dt$$

please see Handout N Derivation of the Fourier series for more info