The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #2 *Solutions (Final Version)*

Date: November 20, 2018   Course: EE 313 Evans

Name: _____ **Spirit Riding Free** _____
                              Last,   First

- The exam is scheduled to last 75 minutes.
- Open books and open notes. You may refer to your homework assignments and the homework solution sets.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- ***Please disable all wireless connections on your calculator(s) and computing system(s).***
- Please power down all cell phones.
- No headphones are allowed.
- All work should be performed on the midterm exam. If more space is needed, then use the backs of the pages.
- **Fully justify your answers.** If you decide to quote text from a source, please give the quote, page number and source citation.

	Problem	*Point Value*	*Your score*	*Topic*
Marciela	1	18		Continuous-Time Convolution
Appoline (Pru)	2	18		Discrete-Time Convolution
Abigail	3	20		Discrete-Time First-Order System
"Lucky"	4	26		Discrete-Time Second-Order System
Miss Flores	5	18		System Properties
	Total	100		

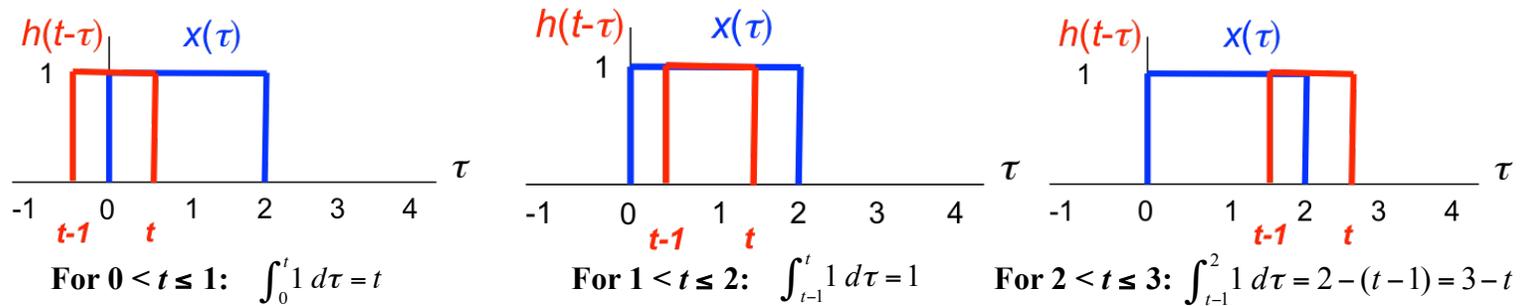**Problem 2.1** *Continuous-Time Convolution.* 18 points.

(a) Plot $y(t) = h(t) * x(t)$ using the rectangular pulse signals below. *9 points.*
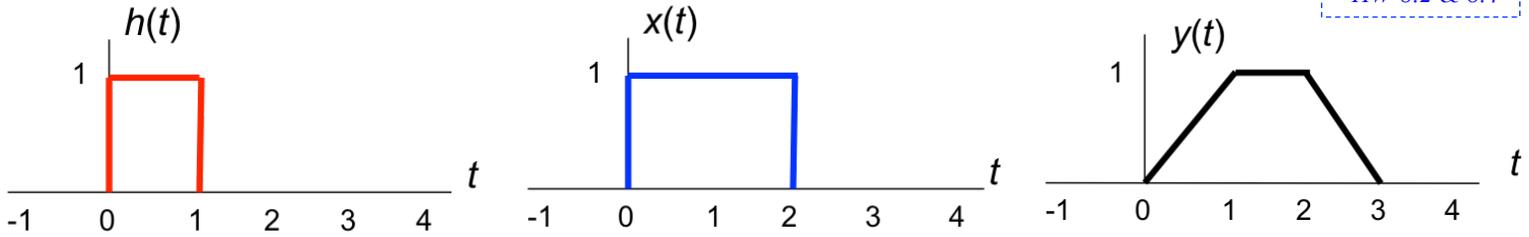
Convolution formula: $h(t) * x(t) \equiv \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$
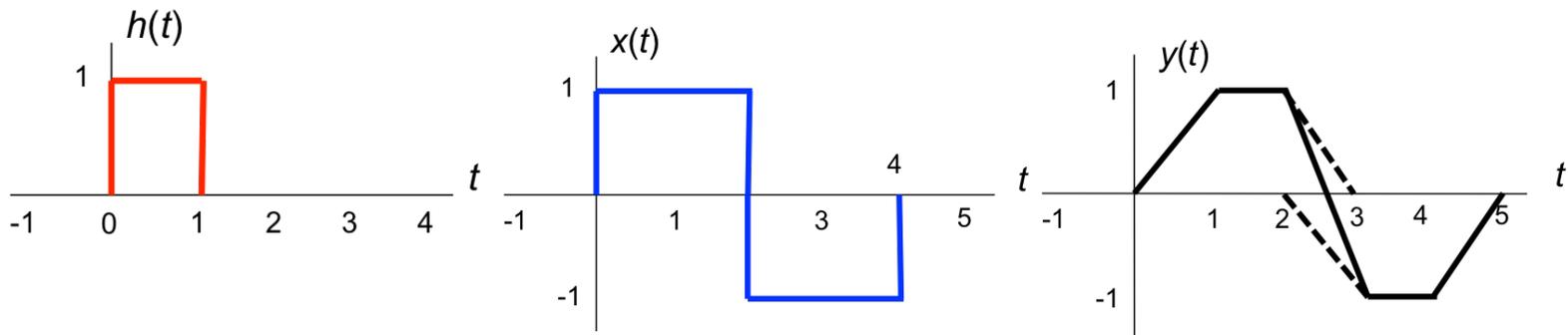
Convolving two causal signals gives a causal result.
Convolving two finite-length signals of lengths L_h and L_x gives result of length $L_h + L_x$.

For $0 < t \le 1$: $\int_0^t 1 \, d\tau = t$ **For $1 < t \le 2$:** $\int_{t-1}^t 1 \, d\tau = 1$ **For $2 < t \le 3$:** $\int_{t-1}^2 1 \, d\tau = 2 - (t-1) = 3 - t$

Convolution of two rectangular pulses of different lengths gives a trapezoid.

(b) Plot $y(t) = h(t) * x(t)$ using the signals below. *9 points*

We can use convolution properties to reuse the result from part (a).
Here $x(t)$ looks like the $x(t)$ in part (a) plus the $x(t)$ in part (a) delayed by 2s and negated.
That is, $x(t) = x_a(t) - x_a(t-2)$.
$h(t) * x(t) = h(t) * (x_a(t) - x_a(t-2)) = h(t) * x_a(t) - h(t) * x_a(t-2) = y_a(t) - y_a(t-2)$.

Application: **Communicating bits in $x(t)$ over an LTI model of a lowpass communication channel with impulse response $h(t)$. Each rectangular pulse of 2 seconds in duration in $x(t)$ carries one bit: amplitude is +1 for a bit of value 1 and -1 for a bit of value 0. If we sample $y(t)$ at 2 seconds and threshold the value against 0, we can recover the first bit of value '1'.**

Problem 2.2 *Discrete-Time Convolution.* 18 points.

(a) Plot $y[n] = h[n] * x[n]$ where $h[n] = \delta[n] + \delta[n-4]$ and $x[n] = \cos(\pi n / 4) u[n]$ such that $u[n]$ is the unit step function. Below, $h[n]$ and $x[n]$ are plotted for $-1 \le n \le 12$, but please keep in mind $x[n]$ is a cosine over $0 \le n < \infty$. *9 points.*

$h[n]$

'U' signal

$x[n]$

n

$h[n]$ could model an acoustic impulse response of sound propagating in a room with one direct path ($n=0$) and one bounce path ($n=4$) from audio speaker to microphone.

$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m] = h[0]\, x[n] + h[4]\, x[n-4]$$

$$y[n] = \cos\left(\frac{\pi}{4}n\right) u[n] + \cos\left(\frac{\pi}{4}(n-4)\right) u[n-4] = \cos\left(\frac{\pi}{4}n\right) u[n] + \cos\left(\frac{\pi}{4}n - \pi\right) u[n-4]$$

$$y[n] = \cos\left(\frac{\pi}{4}n\right) u[n] - \cos\left(\frac{\pi}{4}n\right) u[n-4] = \delta[n] + \frac{1}{\sqrt{2}}\delta[n-1] - \frac{1}{\sqrt{2}}\delta[n-3]$$

$y[n]$

Convolution result is four samples in duration. See the plot on right. Alternately, we can flip and slide $h[n-m]$ about $x[m]$ with respect to m to see that once we have slide $h[n-m]$ so that its leading unit sample is four samples to the right of the origin, we'll add $x[0]$ and $x[4]$ to get zero, $x[1]$ and $x[5]$ to get zero, and so forth.

n

(b) Plot $y[n] = h[n] * u[n]$ using the signals below. Here, $h[n] = -\delta[n] - 4\,\delta[n-1] - 4\,\delta[n-2] - \delta[n-3]$ and is plotted below. Also, $u[n]$ is the unit step function and is plotted below. *9 points.*

'T' signal

$h[n]$

n

$y[n]$

n

$u[n]$

n

$$y[n] = h[n] * u[n] = \sum_{m=-\infty}^{\infty} h[m]\, u[n-m]$$

$$y[n] = -u[n] - 4\,u[n-1] - 4\,u[n-2] - u[n-3]$$

Problem 2.3. *Discrete-Time First-Order LTI IIR System.* 20 points.

Consider a causal discrete-time first-order linear time-invariant (LTI) system with input $x[n]$ and output $y[n]$ governed by the following input-output relationship

$$y[n] - a\, y[n\text{-}1] = x[n] - b\, x[n\text{-}1]$$

for real-valued constants a and b where $|a| < 1$. Input and output signals are observed for $n \geq 0$.

(a) What are the initial conditions? What should their values be? Why? *2 points.*

We compute the first few output samples to reveal the initial conditions:
$$y[0] = a\, y[\text{-}1] + x[0] - b\, x[\text{-}1]$$
$$y[1] = a\, y[0] + x[1] - b\, x[0]$$
Initial conditions are $x[\text{-}1]$ and $y[\text{-}1]$. They must be 0 for LTI properties to hold. That is, a system must be "at rest" as a necessary condition for LTI properties to hold.

(b) Derive the transfer function in the z-domain including the region of convergence. *3 points.*

Take the z-transform of both sides of the difference equation:
$$Y(z) - a\, z^{-1}\, Y(z) = X(z) - b\, z^{-1}\, X(z)$$
$$(1 - a\, z^{-1})\, Y(z) = (1 - b\, z^{-1})\, X(z)$$
Divide both sides by $1 - a\, z^{-1}$ and $X(z)$:
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - b\, z^{-1}}{1 - a\, z^{-1}} = \frac{z - b}{z - a}$$
The z-transform of $a^n\, u[n]$ is $\frac{1}{1 - a\, z^{-1}}$ for $|z| > |a|$.

Region of convergence is $|z| > |a|$ because system is causal (has a causal impulse response).

(c) Give a formula for the frequency response. *3 points.*

Because $|a| < 1$, the region of convergence $|z| > |a|$ includes the unit circle, which allows us to use the transfer function in the z-domain to obtain the frequency response:
$$H_{freq}(\hat{\omega}) = H(z)]_{z=e^{j\hat{\omega}}} = \frac{e^{j\hat{\omega}} - b}{e^{j\hat{\omega}} - a}$$

(d) Give values of a and b for a lowpass filter. Both values of a and b should be close to 1 in absolute value. Make sure the filter is bounded-input bounded-output stable. Justify your choices. *6 points*

Need $|a| < 1$ for BIBO stability. When pole a and zero b are separated in angle, the pole angle indicates passband center frequency and the zero angle indicates stopband center frequency:
$$|H_{freq}(\hat{\omega})| = \left|\frac{e^{j\hat{\omega}} - b}{e^{j\hat{\omega}} - a}\right| = \frac{|e^{j\hat{\omega}} - b|}{|e^{j\hat{\omega}} - a|}$$

For a lowpass filter, the center frequency of the passband is at 0 rad/sample. Let pole $a = 0.9$. The center frequency of the stopband is at π rad/sample. Let zero $b = $ -1.

(e) Give values of a and b for an allpass filter. Both values of a and b should be close to 1 in absolute value. Make sure the filter is bounded-input bounded-output stable. Justify your choices. *6 points*

Using the magnitude response formula in part (d) and Handout I *All Pass Filters*, the pole and zero have to be at the same angle and the zero radius has to be the inverse of the pole radius; i.e., $b = 1/a$ since the pole a and zero b are real-valued. Example: $a = 0.8$ and $b = 1.25$.
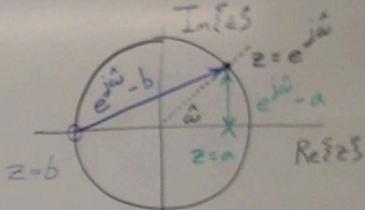
Midterm problem 2.3(d) worked out on a marker board

Midterm Problem 2.3

First-Order IIR Filter Design

Pole is at $z = a$ where $|a| < 1$ for BIBO stability

Zero is at $z = b$

Both a and b are real-valued

$H(z) = \dfrac{z-b}{z-a}$ for $|z| > |a|$

Because $e^{j\hat{\omega}}$ is in the ROC,

$H(e^{j\hat{\omega}}) = \dfrac{e^{j\hat{\omega}} - b}{e^{j\hat{\omega}} - a}$

$|H(e^{j\hat{\omega}})| = \left| \dfrac{e^{j\hat{\omega}} - b}{e^{j\hat{\omega}} - a} \right|$

$|H(e^{j\hat{\omega}})| = \dfrac{|e^{j\hat{\omega}} - b|}{|e^{j\hat{\omega}} - a|}$

$e^{j\hat{\omega}} = \cos(\hat{\omega}) + j\sin(\hat{\omega})$

is a point on the unit circle in 2-D complex plane

$e^{j\hat{\omega}} - b$ is a vector drawn from b to $e^{j\hat{\omega}}$ (ie head-tail)

$z = b$

Magnitude response at $\hat{\omega}$ is

$|H(e^{j\hat{\omega}})| = \dfrac{|e^{j\hat{\omega}} - b|}{|e^{j\hat{\omega}} - a|} = \dfrac{\text{Length of blue vector}}{\text{Length of green vector}}$

For $b = -1$ and $a = 0.9$,

$\hat{\omega}$	$|H(e^{j\hat{\omega}})|$
0	$\dfrac{2}{0.1} = 20$
$\dfrac{\pi}{2}$	$\dfrac{\sqrt{2}}{\sqrt{1.81}} \approx 1$
π	$\dfrac{0}{1.9} = 0$

Lowpass Filter

When the pole and zero are separated in angle and are close to the unit circle, the pole angle determines center of passband and the zero angle determines center of stopband.
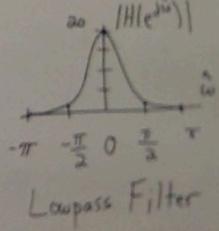
Problem 2.4 *Discrete-Time Second-Order LTI System.* 26 points.

The transfer function in the z-domain for a causal discrete-time second-order linear time-invariant (LTI) system with zeros z_0 and z_1 and poles p_0 and p_1 is given by

$$H(z) = \frac{(z - z_0)(z - z_1)}{(z - p_0)(z - p_1)}$$

We can write the transfer function in negative powers of z as follows:

$$H(z) = \frac{(1 - z_0 z^{-1})(1 - z_1 z^{-1})}{(1 - p_0 z^{-1})(1 - p_1 z^{-1})} = \frac{1 - (z_0 + z_1)z^{-1} + z_0 z_1 z^{-2}}{1 - (p_0 + p_1)z^{-1} + p_0 p_1 z^{-2}} = \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

In the discrete-time domain, the input and output signals are observed for $n \geq 0$.

(a) What is the region of convergence for $H(z)$? *3 points.*

$|z| > \max\{|p_0|, |p_1|\}$ because the system is causal. The region of convergence is the set of all z values for which the z-transform is valid (converges). The z-transform of $p_0^n\, u[n]$ is

$\frac{1}{1 - p_0 z^{-1}}$ for $|z| > |p_0|$. We can apply partial fractions decomposition assuming that $p_0 \neq p_1$:

$$H(z) = \frac{(1 - z_0 z^{-1})(1 - z_1 z^{-1})}{(1 - p_0 z^{-1})(1 - p_1 z^{-1})} = \frac{A_1}{1 - p_0 z^{-1}} + \frac{A_2}{1 - p_1 z^{-1}}$$

The regions of convergence are $|z| > |p_0|$ for the left term and $|z| > |p_1|$ for the right term. A similar result occurs when $p_0 = p_1$.

(b) Derive the difference equation that relates input $x[n]$ and output $y[n]$ in the discrete-time domain. *6 points.*

From $H(z) = \frac{Y(z)}{X(z)} = \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$, we multiply both sides by $1 - a_1 z^{-1} - a_2 z^{-2}$ and $X(z)$:

$(1 - a_1 z^{-1} - a_2 z^{-2})\, Y(z) = (1 + b_1 z^{-1} + b_2 z^{-2})\, X(z)$. Applying the inverse z-transform:

$$y[n] - a_1\, y[n-1] - a_2\, y[n-2] = x[n] + b_1\, x[n-1] + b_2\, x[n-2] \text{ for } n \geq 0$$

$$y[n] = a_1\, y[n-1] + a_2\, y[n-2] + x[n] + b_1\, x[n-1] + b_2\, x[n-2] \text{ for } n \geq 0$$

(c) What are the initial conditions? To what values should the initial conditions be set? *3 points.*

We compute the first few output samples to reveal the initial conditions:

$$y[0] = a_1\, y[-1] + a_2\, y[-2] + x[0] + b_1\, x[-1] + b_2\, x[-2]$$

$$y[1] = a_1\, y[0] + a_2\, y[-1] + x[1] + b_1\, x[0] + b_2\, x[-1]$$

$$y[2] = a_1\, y[1] + a_2\, y[0] + x[2] + b_1\, x[1] + b_2\, x[0]$$

Initial conditions are $x[-1], x[-2], y[-1], y[-2]$. They must be 0 for LTI properties to hold. That is, a system must be "at rest" as a necessary condition for LTI properties to hold.

[Please see the next page for an answer for part (d).]

(d) Choose the values of the two poles and two zeros to design a bandpass filter to pass the fourth octave on the Western music scale and attenuate the other octaves as much as possible. In the fourth octave, the lowest note is at 262 Hz, the highest note is at 494 Hz, and the center frequency is at 378 Hz. Use a sampling rate of 8000 Hz. Your filter design must be bounded-input bounded-output stable. Please plot the poles and zeros on the pole-zero diagram below. *14 points*.

When the poles and zeros are separated in angle, the angle of each pole indicates the frequency of the passband and the angle of each zero indicates the frequency of the stopband.

A bounded-input bounded-output filter must have its poles inside the unit circle. That is, their radii must be less than 1.

**The center frequency of the passband for the filter is**

$$\widehat{\omega}_{pass} = 2\pi \frac{f_{pass}}{f_s} = 2\pi \frac{378\ Hz}{8000\ Hz} = 0.2969\ \text{rad/sample}$$

Pole locations using a radius of $r = 0.9$ to get the poles close to the unit circle. There is a pole for positive frequencies and a pole for negative frequencies:

$$p_0 = r\, e^{j\,\widehat{\omega}_{pass}} \text{ and } p_1 = r\, e^{-j\,\widehat{\omega}_{pass}}$$

Place zeros on unit circle at discrete-time frequencies 0 and π rad/sample, i.e. at $z = \exp(j\,0)$ = 1 and $z = \exp(j\,\pi) = -1$. Humans cannot hear frequencies below 20 Hz, and placing a zero at 0 rad/sample will cause a sharper transition in the magnitude response at 262 Hz.

Not required. **Magnitude response plotted in Matlab in linear units (see the last page for the code).**
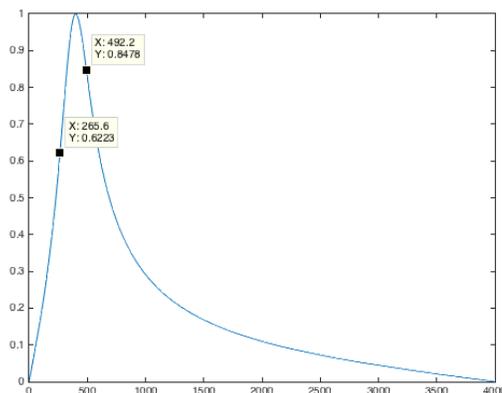
Please see the last page for a comparison of FIR and IIR filter designs to pass the fourth octave on the Western music scale.
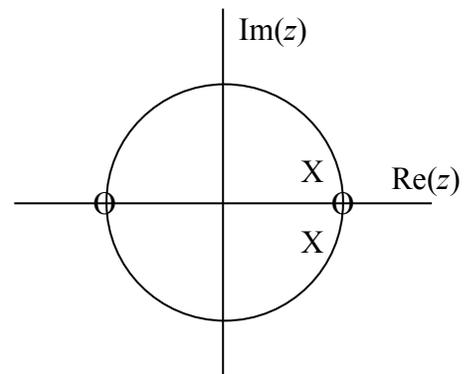
Note: A second-order IIR filter in problem 2.4 and the first-order IIR filter in problem 2.3 are building blocks to create higher-order filters. A fourth-order IIR filter is a cascade of two second-order IIR filters. A fifth-order IIR filter is a cascade of two second-order IIR filters and a first-order IIR filter. And so forth.

Problem 2.5. *System Properties.* 18 points.

Each of the following discrete-time systems has input $x[n]$ and output $y[n]$.

Determine if each system is linear or nonlinear, time-invariant or time-varying, and bounded-input bounded-output (BIBO) stable.

You must either prove that the system property holds in the case of linearity, time-invariance, or BIBO stability, or provide a counter-example that the property does not hold. Providing an answer without any justification will earn 0 points.

Please note that $x[n]$ and $y[n]$ might be complex-valued.

Part	System Name	System Formula	Linear?	Time-Invariant?	BIBO stable?
(a)	Squaring System	$y[n] = x^2[n]$	No	Yes	Yes
(b)	Real part	$y[n] = \text{Re}\{x[n]\}$	No	Yes	Yes
(c)	Running Summation	$y[n] = y[n-1] + x[n]$ for $n \geq 0$	No, since initial condition is not 0	No, since initial condition is not 0	No

For linearity, the properties of homogeneity and additivity must hold.
   Homogeneity: **Input** $\alpha x[n]$. **Output** $y_{\text{scaled}}[n]$. **Does** $y_{\text{scaled}}[n] = \alpha y[n]$ **for all constant** α **values?**
   Additivity: **Input** $x_1[n] + x_2[n]$. **Output** $y_{\text{additive}}[n]$. **Does** $y_{\text{additive}}[n] = y_1[n] + y_2[n]$?
For time-invariance, input $x[n-n_0]$. **Output** $y_{\text{shifted}}[n]$. **Does** $y_{\text{shifted}}[n] = y[n-n_0]$?
For BIBO stability, input $x[n]$. **Is it true that** $|y[n]| < C < \infty$ **for all possible** $|x[n]| < B < \infty$?

(a) Squaring system: $y[n] = x^2[n]$. *6 points.*

   Homogeneity: $y_{\text{scaled}}[n] = (\alpha x[n])^2 = \alpha^2 x^2[n] = \alpha\, y[n] = \alpha\, x^2[n]$ **for all constant** α **values? Only works for** $\alpha = 0$ **and** $\alpha = 1$. **Not homogeneous. Therefore,** *not linear.*
   Time-Invariance: $y_{\text{shifted}}[n] = (x[n-n_0])^2 = x^2[n-n_0] = y[n-n_0] = x^2[n-n_0]$. **Yes. System output only depends on the current input value. Such pointwise systems are time-invariant.** *Yes.*
   BIBO Stability: $|y[n]| = |x^2[n]| = |x[n]|^2 < B^2$. **Here,** $C = B^2$, **and** B **is finite.** *Yes.*

(b) Real part: $y[n] = \text{Re}\{x[n]\}$. *6 points.*
   Homogeneity: **Write scalar** $\alpha = \beta + j\gamma$ **and input signal** $x[n] = r[n] + j\, i[n]$. **The output is**
   $y_{\text{scaled}}[n] = \text{Re}\{\alpha\, x[n]\} = \text{Re}\{(\beta + j\gamma)(r[n] + j\, i[n])\} = \beta\, r[n] - \gamma\, i[n]$
   $y[n] = \text{Re}\{x[n]\} = r[n]$ **and** $\alpha\, y[n] = \beta\, r[n] + j\gamma\, r[n]$.
   Does $y_{\text{scaled}}[n] = \alpha\, y[n]$ **for all constant** α **values? Only when** $x[n]$ **is real-valued.** *No.*
   Time-Invariance: $y_{\text{shifted}}[n] = \text{Re}\{x[n-n_0]\} = y[n-n_0] = \text{Re}\{x[n-n_0]\}$? **Yes. System output only depends on the current input value. Such pointwise systems are time-invariant.** *Yes.*
   BIBO Stability: $|y[n]| = |\text{Re}\{x[n]\}| \leq |x[n]| < B$. **Here,** $C = B$, **and** B **is finite.** *Yes.*

(c) Running summation: $y[n] = y[n-1] + x[n]$ for $n \geq 0$. *6 points.*

   Linearity & Time-Invariance: **Necessary condition is for the system to be at rest. System has one initial condition at** $y[-1]$ **whose value is not specified. Neither linear nor time-invariant.**
   BIBO Stability: **Let** $y[-1] = 0$ **and** $x[n] = u[n]$. $y[n] = (n+1)\, u[n]$ **which is unbounded as** n **tends to infinity.** *No.* **(For** $x[n] = u[n]$, **general solution is** $y[n] = (n+1)\, u[n] + y[-1]\, (n+1)\, u[n]$.)

```matlab
% MATLAB Code

% Code for problem 2.2(a)
n = -1:12;
h = zeros(1,14);
h(2) = 1;
h(6) = 1;
stem(n, h);
ylim( [-1.2 1.2] );
xlim( [-1.5 12.5] );

n = -1:12;
x = cos(pi*n/4);
x(1) = 0;
stem(n, x);
ylim( [-1.2 1.2] );
xlim( [-1.5 12.5] );
title( 'x[n]' );

y = zeros(1,14);
n = 0:3;
y(2:5) = cos(pi*n/4);
n = -1:12;
stem(n, y);
ylim( [-1.2 1.2] );
xlim( [-1.5 12.5] );

% Code for problem 2.2(b)
n = -1:12;
y = -stepfun(n,0) - 4*stepfun(n-1,0) - 4*stepfun(n-2,0) - stepfun(n-3, 0);
stem(n, y);
ylim( [-10.2 1.2] );
xlim( [-1.5 12.5] );

% Code for problem 2.3
figure; freqz( [1 1], [1 -0.9] );       % (d) lowpass filter
figure; freqz( [1 -1.25], [1 -0.8] );   % (e) allpass filter

% Code for problem 2.4(d) bandpass filter
fpass = 378;        % center frequency of the 4$^{th}$ octave on the Western scale
fs = 8000;          % sampling rate

z0 = 1;
z1 = -1;
numer = [1 -(z0+z1) z0*z1];

r = 0.9;
poleAngle = 2*pi*fpass/fs;
p0 = r * exp(j*poleAngle);
p1 = r * exp(-j*poleAngle);
denom = [1 -(p0+p1) p0*p1];

gain = 0.095;    % normalizes maximum magnitude response to 1 in linear units

[H, W] = freqz(gain*numer, denom);
F = W*fs/(2*pi);
plot(F, abs(H));
```

Problem 2.4(d). *Not required.* **Design of a bandpass filter.**

We'll take a different look at the design of a bandpass filter to pass the fourth octave on the Western scale than mini-project #2 took.

In our case, we seek to design a bandpass filter that meets the following specification:

-40 dB stopband attenuation (Astop1) from 0 to 247 Hz,
- 1 dB to 0 dB passband response (Apass) from 262 to 494 Hz, and
-40 dB stopband attenuation (Astop2) at and above 523 Hz.

The sampling rate is 8000 Hz.

We'll use a MATLAB filter design interface called filter design and analysis tool (fdatool).

FIR Filter Design: Type fdatool in MATLAB. Select Bandpass for "Response Type". Select "FIR Filter" for Design Method and then select "Equiripple" for the specific FIR Filter Design Method. Then type in the above design specifications (except use 1.5 dB for Apass which is workaround for a limitation of the design method implementation) and hit "Design Filter". The resulting design is a 690th order FIR filter, which has 691 coefficients and requires 691 multiplications/additions to compute one output sample.

IIR Filter Design: Type fdatool in MATLAB. Select Bandpass for "Response Type". Select "IIR Filter" for Design Method and then select "Elliptic" for the specific IIR Filter Design Method. Then type in the above design specifications and hit "Design Filter". The resulting design is a 12th order IIR filter, which has 25 coefficients and requires 25 multiplications/additions to compute one output sample.

The IIR filter is 27 times more efficient in computation, and hence, IIR filters are more commonly used in digital audio applications than FIR filters.
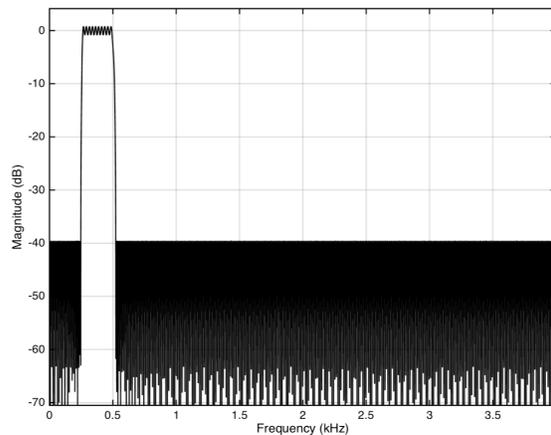
Magnitude Response for the
FIR Bandpass Filter Design
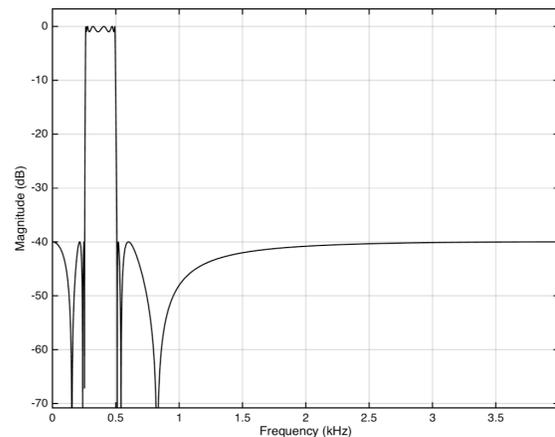
Magnitude Response for the
IIR Bandpass Filter Design