Algorithm for symbolic design of elliptic filters

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Abstract— **A new algorithm for symbolic design of analog and digital IIR elliptic (Cauer) filters is presented. The algorithm is based on the nesting feature of elliptic functions. The advantages of the proposed algorithm are illustrated by several examples.**

I. Introduction

A classical elliptic filter design algorithm relies on numerically oriented procedures. Many simple approximate algorithms exist for those numerical design algorithms [1] - [4]. In general, these numerical design procedures lead to only one solution, which is often far from the optimum for the given constrains or design goals. Even exhaustive and repetitive numerical calculations can fail in finding the best solution.

This paper presents a symbolic algorithm implemented in a computer algebra system to design and synthesis elliptic filters, including the Chebyshev and Butterworth types as special cases. Symbolic design make it possible to eliminate redundant variables, to decrease the order of the functions, and to simplify or approximate the complex relations prior to the final numerical calculations. Closed-form expressions are preferred and maintained in symbolic form up to the point where numerical evaluation is ultimately necessary. The benefits of this approach are clarified by several examples. The original algorithm has been developed in Mathematica [5].

II. General algorithm for elliptic filter symbolic design

Without lack of generality let us consider a low-pass filter design. The filter specification of another type (high-pass, bandpass, bandreject) is transformed into the equivalent low-pass prototype that meets the specification

$$S = \{F_p, F_s, A_p, A_s\} \tag{1}$$

where F_p, F_s, A_p and A_s are the pass-band edge frequency, the stop-band edge frequency, the maximum pass-band attenuation and the minimum stop-band attenuation in dB, respectively.

The first step, i.e. approximation, is to generate a characteristic function $K(f)$ that satisfies the desired specification S. The attenuation, A, can be expressed in terms of $K(f)$

$$A(f) = 10 \log \left(1 + K^2(f)\right) \tag{2}$$

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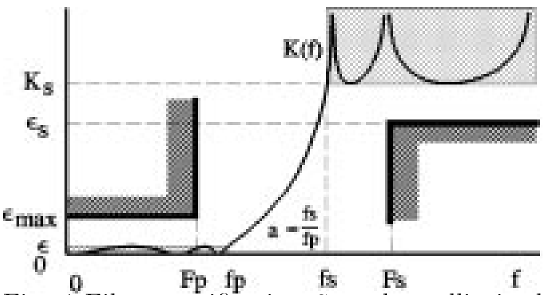

Fig. 1 Filter specification S_ϵ and an elliptic characteristic function $K(f)$.

There is an infinite number of characteristic functions that fit S. In order to find the better relation between S and $K(f)$, we will consider the mapped specification S_ϵ

$$S_\epsilon = \{F_p, F_s, \epsilon_{max}, \epsilon_s\} \tag{3}$$

where

$$\begin{aligned} \epsilon_{max} &= \sqrt{10^{A_p/10} - 1} \\ \epsilon_s &= \sqrt{10^{A_s/10} - 1} \end{aligned} \tag{4}$$

as shown in Fig.1.

In practice, the filter is implemented with non-ideal elements that have finite tolerances. In order to get the higher manufacturing yield using larger tolerances a characteristic function should be determined with safety margins so that the actual $K(f)$ satisfies the new specification list

$$S_K = \{f_p, f_s, \epsilon, K_s\} \tag{5}$$

as shown in Fig. 1 for an elliptic filter.

The Chebyshev, inverse Chebyshev and Butterworth types are included in (5) as special cases for $\{f_s \to +\infty\}$, $\{f_p \to 0\}$ and $\{f_s \to +\infty, f_p \to 0\}$, respectively.

A filter design implies finding the design list D for a given S

$$D = \{f_p, a, \epsilon, n\} \tag{6}$$

where f_p is the pass-band edge frequency, a is the selectivity factor

$$a = \frac{f_s}{f_p} \tag{7}$$

ϵ is the pass-band ripple and n is the order of the desired filter.

The elliptic characteristic function $K(f)$ is uniquely defined by D, with $\omega = 2\pi f/f_p$

$$K(f) = \epsilon |R(n,\omega,a)|, \begin{cases} |R(n,\omega,a)| \le 1, & |\omega| \le 1 \\ |R(n,\omega,a)| \ge R(n,a,a)|, & |\omega| \ge a \end{cases} \tag{8}$$

while S determines only the boundary values

$$\left\{ \begin{array}{ccccc} 0 & \leq & f_p & < & F_s \\ a_{min} & \leq & a & \leq & a_{max} \\ \dfrac{\epsilon_s}{R\left(n, \dfrac{F_s}{F_p}, \dfrac{F_s}{F_p}\right)} & \leq & \epsilon & \leq & \epsilon_{max} \\ n_{min} & \leq & n & & \end{array} \right\} \quad (9)$$

where a_{min} and a_{max} can be found from:

$$\left\{ \begin{array}{ll} R(n, a, a) = \dfrac{\epsilon_s}{\epsilon_{max}} & \rightarrow \quad a_{min} \\ R\left(n, \dfrac{F_s}{F_p}, a\right) = \dfrac{\epsilon_s}{\epsilon_{max}}, a > \dfrac{F_s}{F_p} & \rightarrow \quad a_{max} \end{array} \right. \quad (10)$$

$R(n, \omega, a)$ denotes the nth-order rational elliptic function. It is known that the ordinary elliptic function provides the minimal order n_{min} for the given S [3].

Note that ϵ_{max} depends only on A_p while ϵ_{min}, a_{min} and a_{max} are functions of n

$$\begin{array}{ccccc} \epsilon_{min}(n) & > & \epsilon_{min}(n+1) & > & 0 \\ a_{min}(n) & > & a_{min}(n+1) & > & 1 \\ a_{max}(n) & < & a_{max}(n+1) & < & +\infty \end{array} \quad (11)$$

In purely numerical procedures, the computation cost is usually very high because the approximation step may have to be repeated many times. Even exhaustive and repeated calculations can fail in finding the best solution. In general, the designer prefers an approximation that yields a design with the minimal filter order. On the contrary, it has been reported [6], [10]-[12] that higher-order analog and digital elliptic filters may be much more efficient than the minimal-order design commonly used in only-numerical procedures.

It is computationally expensive to solve a set of nonlinear equations with four independent variables of D. It is practically impossible to simplify this set of equations by hand. The only acceptable solution is automated symbolic computation.

The design can proceed with symbolic expressions. The filter transfer function can be expressed in terms of poles, s_i and attenuation zeros, $\omega_i(n, a)$:

$$H(s) = \prod_{i=1}^{n} \frac{s - j\,\dfrac{a}{\omega_i(n, a)}}{s - s_i}, \qquad i = 1, ..., n \quad (12)$$

where

$$\begin{array}{rcl} s_i & = & \sigma_i + j\Omega_i \\ \sigma_i & = & \dfrac{-\zeta\sqrt{1-\zeta^2}\sqrt{1-\omega_i^2(n,a)}\sqrt{1-\dfrac{\omega_i^2(n,a)}{a^2}}}{1-\zeta^2\left(1-\dfrac{\omega_i^2(n,a)}{a^2}\right)} \\ \Omega_i & = & \dfrac{\omega_i(n,a)\sqrt{1-\left(1-\dfrac{1}{a^2}\right)\zeta^2}}{1-\zeta^2\left(1-\dfrac{\omega_i^2(n,a)}{a^2}\right)} \\ \zeta & = & \dfrac{1}{\sqrt{1+E^2(n,a,\epsilon)}} \end{array} \quad (13)$$

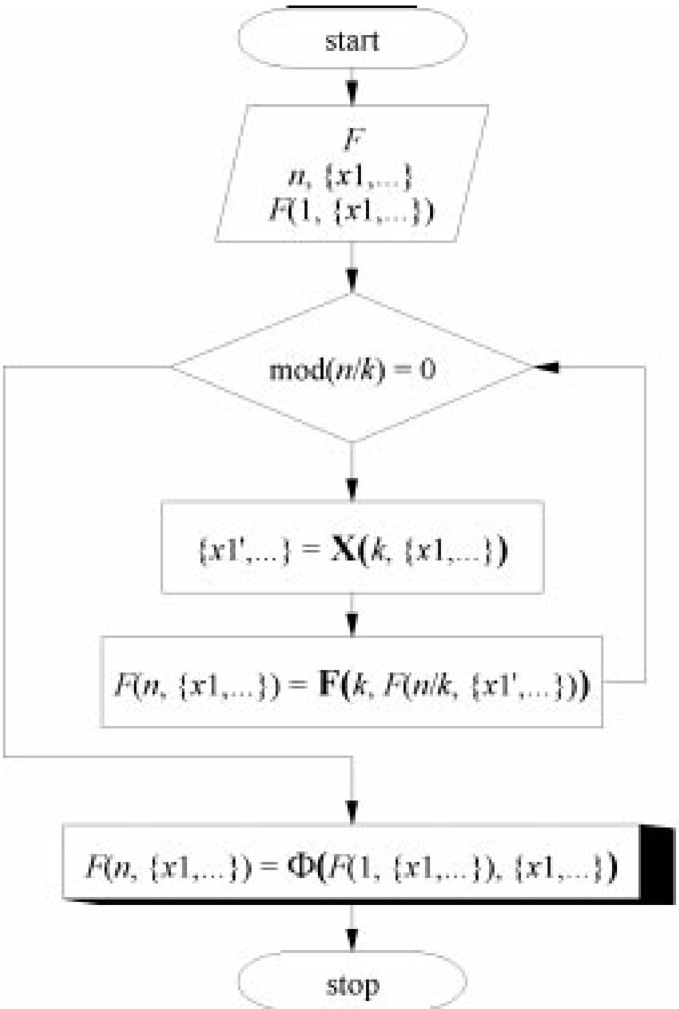

Fig. 2 General algorithm for symbolic synthesis of elliptic filters.

A new function $E(n, a, \epsilon)$ is introduced to evaluate ζ [7].

The functions $R(n, \omega, a)$, $\omega_i(n, a)$ and $E(n, a, \epsilon)$ are closed-form symbolic expressions in terms of the design list D. Thus, the transfer function poles and zeros are also symbolic expressions in terms of n, a and ϵ, i.e. D.

The general algorithm for evaluation of $R(n, \omega, a)$, $\omega_i(n, a)$ and $E(n, a, \epsilon)$ is given in Fig. 2. The design parameters a and ϵ are given by symbols. Using the nesting feature of elliptic functions [7], [8] the higher-order functions are symbolically obtained from the lower-order ones, by a successive recursive application of the algorithm from Fig. 2. The order is decreased until the unit value, for which the functions are known.

The main loop executes until the remainder of n/k is not zero. The quantity k is a prime number greater than one and less than n. For $k = 2$ and $k = 3$ analytic closed-form symbolic expressions are used [7],[8]. For the other prime k values, k= 5, 7, 11, ..., the Jacobi elliptic functions can be used. Currently, the research efforts are directed towards the case k=5 and finding the solution free of the Jacobi functions.

In Fig. 2, the operator $\mathbf{X}$ designates a procedure for symbolic generation of modified and intermediate parameters (designated by $x1,...,x1',...$). The operator **F** stands for the procedures that evaluate $R(n,\omega,a)$, $\omega_i(n,a)$ and $E(n,a,\epsilon)$ in the symbolic form. The operator Φ means that all filter quantities (poles, zeros, transfer function, pole-Q factors, multiplier coefficients, ...) can be expressed in closed forms (symbolically) in terms of D.

The proposed algorithm has been fully implemented and tested in Mathematica on a PC platform.

III. Applications of filter symbolic design

A new symbolic algorithm is successfully applied in design centering and tolerance analysis. For the given specifications the minimal or prescribed values of Q-factors and minimal number of multipliers in digital filter are calculated.

Example 1. Consider a design of a low-pass elliptic filter specified by

$$S = \{F_p = 1\text{kHz}, F_s = 1.075\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$$

Let us find the minimal filter order n and the maximal Q-factor from the prescribed manufacturer's values [9]:

$$Q_i \in \{..., 22.6, 30.2, 32.0, ...\}$$

The center frequency can take a value from the list [9]:

$$\frac{|s_i|}{2\pi F_p} \in \{..., 1.0210, 1.0108, 1.0053, ...\}$$

The minimal filter order can be calculated according to [4]. The domain of the parameters a and ϵ are found from (9) for $n=8$:

$$\left\{\begin{array}{ccccc} 0 & \leq & f_p \approx F_p & < & F_s \\ 1.04285 & \leq & a(n=8) & \leq & 1.08323 \\ 0.09097 & \leq & \epsilon(n=8) & \leq & 0.217 \\ 8 & \leq & n & & \end{array}\right\} \tag{14}$$

The filter is designed as a cascaded connection of the second-order sections. For each second-order filter section, $|s_i|/2\pi F_p$ and Q_i are chosen independently from the corresponding manufacturer's lists [9]. The actual center frequency is a function of the filter's clock rate, 6-bit control word for $|s_i|/2\pi F_p$ [9, pp. 6.22 - 6.23] and operating mode. The Q_i of each section is also set by a separate programed input [9, pp. 6.23 - 6.24]. This way, each second-order filter section is tuned independently.

Using the symbolic design strategy, closed-form expressions for the maximal Q-factor, Q_1, and corresponding $|s_1|/2\pi F_p$ are derived first, in terms of ω_1 and ζ. Next, ω_1 can be eliminated and only one nonlinear equation, relating Q_1, $|s_1|/2\pi F_p$ and ζ, remains. Choosing Q_1 and $|s_1|/2\pi F_p$ from the lists of available values, and solving the equation for a and ϵ (from the design domain): a=1.0559, ϵ=0.145562, Q_1=32.0, $|s_1|/2\pi F_p$=1.0108. These values meet the specification.

Let us now review the classical, purely numerical, approach and show its drawbacks. It starts by choosing a=F_s/F_p=1.075 and ϵ=ϵ_{min} = 0.09097, when $Q_{1,max}$=24.24 and corresponding $|s_1|/2\pi F_p$=1.0193. Since these values are unavailable, the adjacent values are taken. The pass-band ripple factor obtained is greater than 0.7dB and is far from the required value, implying failure of the design.

Table 1 F_p=1kHz, F_s=1075Hz, A_p=0.2 dB, A_s=40 dB

test	type	n	Q_{max}	$A(F_p)$	$A(F_s)$
1	Butterworth	85	-		
2	Chebyshev	18	46		
3	Inverse Chebyshev	18	20		
4	elliptic	8	24		
	symbolic design	12	16	0.036	42.6

Example 2. Find the minimal filter order n and the minimal value of the maximal Q-factor ($Q_{max} \leq 20$) for the same S given in example 1.

Usually, the classical approximations are tried first. The design results are summarized in Table 1. Obviously, the four classical designs failed to meet requirements: n is too high or $Q > 20$.

By using the same symbolic design as in example 1, the requirements are successfully met. According to [10]

$$Q_i = \frac{a + \omega_i^2}{2\sqrt{(1-\omega_i^2)(a^2-\omega_i^2)}} \tag{15}$$

for $a = F_s/F_p$, Q_{max} has been found to be $Q_{max} = 16$ for $n = 6$. Cascading two 6th-order elliptic filters, the specification S is fulfilled.

Example 3. The requirements are taken from [2], example No. 4: sampling frequency F_0=16kHz and specification

$$S = \{F_p = 3.4\text{kHz}, F_s = 4.6\text{kHz}, A_p = 0.2\text{dB}, A_s = 65\text{dB}\}$$

The 7th-order elliptic IIR filter, realized with 7 multipliers, was used in [2] to fulfill the requirements. In [11], in a few trials, it was shown that the requirements are fulfilled with only 3 multipliers (β_2=0.2985, β_3=0.8432, β_4=0.0858) and 6 shifters ($\alpha_1 = -1/2^5$, $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha = -1/2^4$, β_5= $1/2 + 1/2^4$) with f_p=3.12kHz and f_s=4.574kHz. In spite of that $f_p = 3.12$kHz is smaller than F_p=3.4kHz, the maximal attenuation at the frequency F_p is substantially below A_p, i.e. $A(0 \leq f \leq F_p) < 0.004$dB, $A(f \geq F_s) > 66.7$dB.

The target of this design is to implement a multiplier with one adder and two shifters. It requires that the coefficients should have values $b_m \in \{1/2^i \pm 1/2^j\}$ (i, j integers). For $f_s = 4.6$kHz it was found that β_5=0.5565 is the closest to $1/2 + 1/2^4$. The underlining idea of the symbolic design is to express β_5 in terms of a; next a is found from $\beta_5(a) = 1/2 + 1/2^4$.

The coefficient β_5 is the pole magnitude squared, in z plane,

$$\beta_5 = \frac{k^2 + 2k\sigma_5 + a}{k^2 - 2k\sigma_5 + a} \tag{16}$$

obtained by the bilinear transformation

$$s = k\frac{z-1}{z+1}$$

with the transformation constant k

$$k = \sqrt{a\ \frac{1-\alpha}{1+\alpha}} \tag{17}$$

while σ_5, given in (13), can be expressed in a simplified form as shown in [10]:

$$\sigma_5 = -\frac{\sqrt{a(1-\omega_5^2)(a-\omega_5^2)}}{a+\omega_5^2} \tag{18}$$

The corresponding ω_5 can be expressed only in terms of a as shown in [8]. Therefore, β_5 is determined by a single nonlinear function in a single variable a. Now, from the equation $\beta_5(a) = 1/2 + 1/2^4$ we can determine a by using standard numerical methods.

Note, that without symbolic simplification, a set of nonlinear equations in three variables ϵ, a and f_p must be solved. A simplification by hand is not possible because of the complexity of the expression $\beta_5(a)$.

The proposed symbolic approach enables a further simplifications. The rather involved function, $\beta_5(a)$ is approximate by the Taylor polynomial at point $a = a_{max}$ [11]

$$a_{max} = \frac{1-\alpha}{1+\alpha}\ \tan^2 \pi\frac{F_s}{F_0} = 1.82362 \tag{19}$$

while $\alpha{=}-1/2^4$ has been fixed by the design [11]. Finally, we have a simple expression for β_5

$$\begin{aligned}\beta_5(a) = \quad & 0.556498 - 0.155813(a-a_{max}) \\ & +0.114708(a-a_{max})^2 + 0.0886438(a-a_{max})^3\end{aligned} \tag{20}$$

The solution of equation $\beta_5(a) = 1/2 + 1/2^4$ is straightforward: $a{=}1.78616$.

The above symbolic algorithm yields a simple polynomial analytic expression for $\beta_5(a)$. If the preferred values for β_5 should be changed, the new a can be determined in the same manner by using (20). On the contrary, the traditional numerical procedures must reevaluate all steps from the very beginning of the design procedure.

IV. Remarks on symbolic design

Symbolic design approach made possible efficient realizations, sharper amplitude characteristics, nearly maximally flat pass-band and smaller radii of the poles in parallel realizations [12]. The proposed symbolic algorithm finds digital filters having one half of the coefficients exactly equal to zero. Closed-form relations eliminate certain kind of numerical errors, unavoidable in purely numerical algorithms. In the example from [12], by using numerical approach, the calculated coefficients are very small, but different from zero. Usually, filter designers discard such a realization due to increased number of nonzero coefficients (although a half of the coefficients are nonzero due to numerical errors).

Generally, from the calculated poles and zeros, the filter coefficients are simply obtained by rounding or truncating to the available number of bits, or they have to be assigned to the nearest realizable values. A more complicated alternative is to consider finding the design list D as a problem of discrete optimization. This way designers choose the realizable pole and zero values that best fit into the specification S. This rather involved approach is considerably simplified by the symbolic algorithm as shown in example 3.

V. Conclusion

This paper brings into focus a new general algorithm for symbolic design of elliptic filters. The characteristic function, the transfer function poles and zeroes of elliptic filters are found as closed-form symbolic expressions in terms of two design parameters: (1) the stop-band edge to pass-band edge frequency ratio, and (2) the ripple factor. Using the nesting feature of elliptic functions higher-order characteristic functions are obtained by a successive recursive application of the new algorithm introduced. The new nesting properties are found and exploited for the symbolic evaluation of poles and zeros.

The design algorithm proceeds symbolically until the technological requirements are given. Symbolic synthesis can give more insight into the influence of filter specification on the design parameters and actual, designed, Q-factors, poles, zeros, multiplier coefficients, etc.

The advantages of the symbolic design are illustrated by several examples. It has been shown that the symbolic design of analog and digital filters is much better than the best known purely numerical design.

The original algorithm has been developed and tested in Mathematica.

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