Design Space Approach to Advanced Filter Design

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Abstract— **This paper reviews the basic definitions of a filter design. It introduces straightforward procedures to map the filter specification into the design space, i.e. a set of ranges for parameters that we use in the filter design. We search this design space for the optimum solution according to given criteria, such as minimal Q-factor or minimal quantization error.**

The principal drawback of the classical filter design is in returning only one solution, which can be unacceptable for many practical implementations. We propose a new approach to the filter design by using a mixture of symbolic and numeric computation and discrete non-linear optimization. This approach should provide reduced filter complexity for a desired performance, or better performances for the required complexity.

I. Introduction

A *filter* is a system, that can be used to modify, reshape or manipulate the frequency spectrum of an analog or digital signal according to some prescribed requirements.

An *electrical filter* may be used to amplify or attenuate a range of frequency components (sinusoidal signals) or to reject or isolate one specific frequency component. The applications are numerous: to eliminate signal contamination such as noise in communication systems, to separate relevant from irrelevant frequency components, to detect signals in radios and TV's, to demodulate signals, to bandlimit signals before sampling, to convert sampled signals into continuous-time signal, to improve quality of audio equipment, in time-division to frequency-division multiplex systems, in speech synthesis, in the equalization of transmission lines and cables, in the design of artificial cochlesas [1] in audio, video, speech, voiceband modems, control, instrumentation, radio signaling and radar, high definition television, radio modems, seismic modeling, financial modeling, weather modeling.

A *digital filter* takes an input sequence of numbers and produces an output sequence of numbers. Usually, the input sequence of numbers are samples of a continuous function of time; but, it can be any kind of numbers such as prices from daily stock market, pixels of image, ...

Generally, the purpose of most filters is to separate the desired signals from undesired signals or noise. Often, the descriptions of the signals and noise are given in terms of their frequency content or the energy of the signals in the frequency bands. For this reason, the filter specifications are usually given in the frequency domain as magnitude response or by gain or attenuation.

The range of frequencies in which the sinusoidal signals are rejected is called *stopband*. The range of frequencies in which the sinusoidal signals pass with tolerated distortion is called *passband*. A region between the passband and stopband, where neither desired nor undesired signals exist or the spectra of the desired and undesired signals are overlapped, can be defined as a *transition region*.

In this paper we will consider a filter with single passband referred to as *low-pass* filter. All other types of filters (*high-pass*, *bandpass*, and *bandstop*) can be easily obtained by simple transformation from the low-pass filter.

Once the filter requirements are known, the filter specification can be established, e.g. we specify the passband and stopband edge frequencies and tolerances. Next, we proceed with the filter design.

The *design* is a set of processes, that starts with the specification, and ends with the implementation of a (product) filter prototype. It comprises four general steps, as follows: 1) approximation, 2) realization, 3) study of imperfections, 4) implementation.

The *approximation* step is the process of generating a transfer function that satisfies the desired specification.

The *realization* step is the process of converting the transfer function of the filter into a network or a set of equations.

The *study of imperfections* investigates the effects of element imperfections, which determines the highest tolerance that can be tolerated without violating the specification of the filter throughout its working life.

The *implementation* step is constructing the product prototype of the filter in hardware or software. Decisions to be made involve the type of components and packaging, and the methods to be used for the manufacture, testing, and tuning of the filter, the data and coefficient wordlengths, etc.

Usually, those four design steps are considered separately, although they are not independent of each other. The main goal is to find the most economical solution in short time. Which filter is better depends on the hardware used for the implementation. Many different constrains have to be fulfilled. The finite word-length effects, component tolerances, parasitic effects, have significant influence on fulfilling the specification. In this case, classical approaches are not adequate for optimizing both the behavior (performance) and implementation (complexity and cost).

We propose a new approach to the filter design by us-

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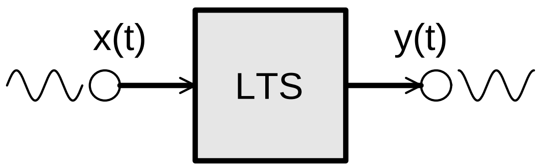

Fig. 1. Linear Time-invariant System.

ing a mixture of symbolic and numeric computation and discrete non-linear optimization. Opposite to the conventional approaches, that return only one design and hide a wealth of alternative filter designs, the advanced design techniques (that we introduce) find a comprehensive set of optimal designs to represent the infinite solution space.

II. Specification and basic definitions

Usually, the filter specification is given, but in many cases the designer has to establish the specification by himself. This is the most important prerequisite for the filter design. Namely, if the specification is too restrictive (e.g. very low passband and stopband tolerances, narrow transition region), the filter may not be feasible.

The special care must be taken in determining the passband and stopband tolerances. For example, if the noise due to product quantization, or the noise at the output of an amplifier, is -60 dB, it will be not reasonable to require 80 dB attenuation in stopband. The generated noise in the filter will be much higher than the attenuated undesired signal.

When down-sampling is performed in a digital filter, the lower or higher half of the spectra have to be rejected in order to prevent aliasing effects. In many cases just 20 or 30 dB attenuation in stopband will be sufficient.

The proper selection of the specification must be done according to the nature of signals (i.e. frequency bands and the corresponding levels of the desired and undesired signals or noise) and the available hardware or software (floating point or fixed point arithmetic, element tolerances and parasitic effects, etc.).

In this paper we assume that the specification is given. Next, we examine the feasibility of the practical filter design. Finally, if the filter is not feasible, we propose the minimum changes in the given specification to make the filter design possible.

In practice, there are several ways in presenting specifications. Usually, designers of analog filters prefer attenuation or gain expressed in dB, while magnitude tolerances are more convenient for the designers of digital filters.

To provide a unified and consistent design we adopt one form of presenting the specification.

Let us consider a *linear time-invariant system* (LTS), with the input sine signal $x_{in}(t)$

$$x_{in}(t) = X_m \sin(\omega t + \xi) \tag{1}$$

and the output signal $y_{out}(t)$,

$$y_{out}(t) = Y_m \sin(\omega t + \eta) \tag{2}$$

as indicated in Fig. 1. With $M(\omega) = Y_m/X_m$ we can describe the change in magnitude and we call $M(\omega)$ the *magnitude response*; with $\theta(\omega) = \eta - \xi$ we designate the change in phase and we call $\theta(\omega)$ the *phase response*; both quantities are defined at the frequency $\omega = 2\pi f$ from the frequency range of interest. The *frequency response* of the system is defined as $H(\omega) = M(\omega)e^{j\theta(\omega)}$ and uniquely describes both, an analog or a digital filter.

In other words, $H(\omega)$ shows how the input signal is transferred through the system at the specific frequency ω rad/s. From the frequency response we can derive the *transfer function* H. In analog filter theory $H = H(s)$ is in terms of the complex frequency $s = j\omega$. In digital filter theory $H = H(z)$ is in terms of $z = e^{j\omega T}$, where T designates the sampling period. H is a rational function of s or z. The magnitude of the transfer function, H, for real frequencies, ω, is $M(\omega) = |H|$.

Several functions are derived from the magnitude response and are frequently used in practice. The reciprocal of the squared magnitude is called the *loss function*

$$L_F(\omega) = \frac{1}{M^2(\omega)} \tag{3}$$

We will call the function $\sqrt{(L_F(\omega) - 1)}$ the *characteristic function*

$$K(\omega) = \sqrt{\frac{1}{M^2(\omega)} - 1} \tag{4}$$

Attenuation (in dB) or *loss characteristic* is defined by

$$A(\omega) = 20 \log_{10} \frac{1}{M(\omega)} \tag{5}$$

Gain (in dB) is the negative of attenuation

$$G(\omega) = -A(\omega) = 20 \log_{10} M(\omega) \tag{6}$$

In this paper we assume that the maximal value of the magnitude is 1, $M(\omega)|_{max} = |H|_{max} = 1$. If this is not the case, the required attenuation or gain can be easily compensated for by multiplying the digital signal by a constant, or by a frequency independent amplifier.

In Fig. 2 the magnitude of a low-pass filter is shown in terms of frequency. In case of a digital filter the upper limit frequency is half the sampling frequency because we assume that the signal spectrum does not exist at higher frequencies. In case of an analog filter, theoretically, no upper limit frequency exists; however, we can assume that the practical upper frequency is up to 100 times higher than the maximal frequency of interest.

The filter specification can be expressed in several ways. (1) The magnitude limits, Fig. 2, define the minimum magnitude in passband, M_p, and the maximum magnitude in stopband, M_s. (2) The magnitude tolerances, Fig. 3, specify the maximum magnitude decrease in passband,

$\delta_p = 1 - M_p$, and the maximum magnitude in stopband, $\delta_s = M_s$. (3) The magnitude ripple tolerance, Fig. 4, describes the maximum magnitude variation, in passband, δ_1, and in stopband, δ_2. (4) The attenuation limits in dB, Fig. 5, specify the maximum attenuation in passband, A_p, and the minimum attenuation in stopband, A_s. (5) The gain limits in dB, Fig. 6, specify the minimum gain in passband, $G_p = -A_p$, and the maximum gain in stopband, $G_s = -A_s$.

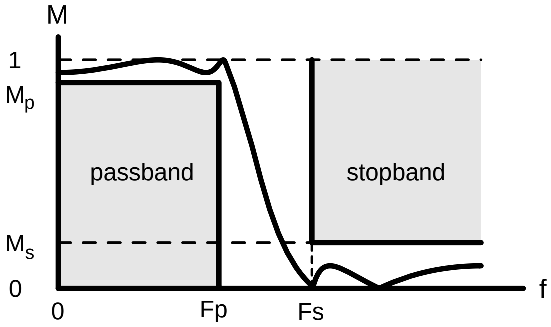

Fig. 2. Magnitude limit specification.

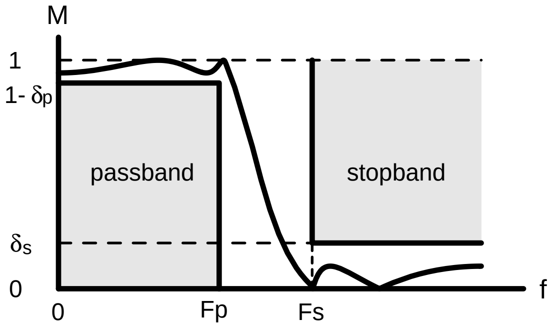

Fig. 3. Magnitude - tolerance specification.

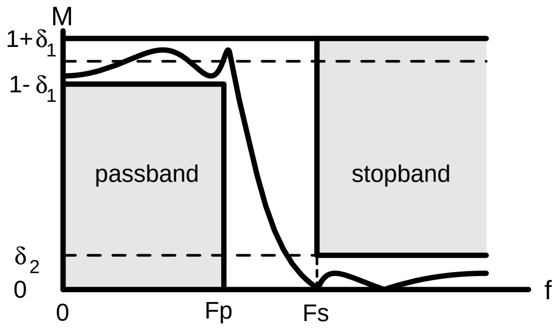

Fig. 4. Magnitude ripple specifications.

Relations between the specification quantities are summarized in Table 1.

The underlining idea of the design that we propose in this paper is to map the specification into a new one, ex-

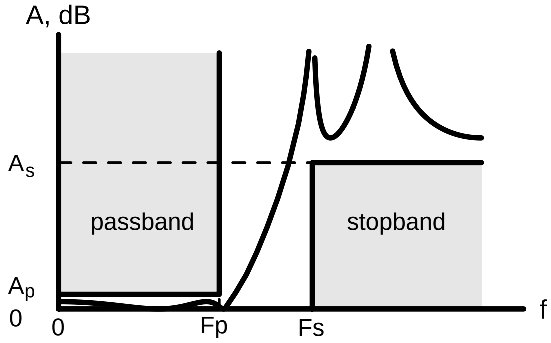

Fig. 5. Attenuation limit specification.

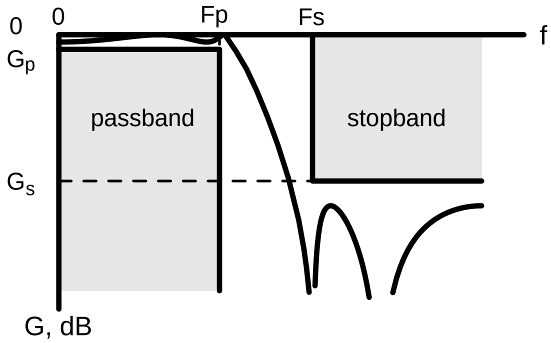

Fig. 6. Gain limit specification.

pressed in terms of the characteristic-function limits, Fig. 7, specifying the maximum value in passband, K_p, and the minimum value in stopband, K_s. This provides a unified start for the subsequent design steps.

III. Approximation problem

In this section we consider the approximation step of the design process.

Let us consider the specification shown in Fig. 7. The first step is to generate an auxiliary function in ω, that is called the *approximation function* (or *approximation*), from which the transfer function meeting the specification can be derived. Besides several classical approximations

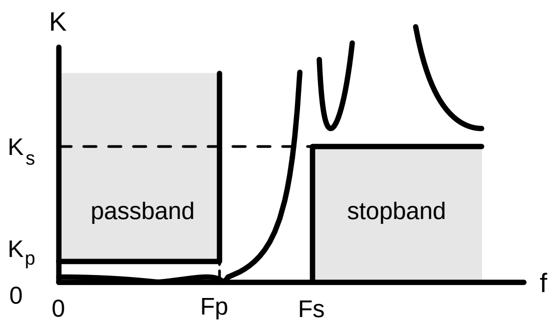

Fig. 7. Characteristic function limit specification.

Table 1. Relations between specification quantities.

magnitude		magnitude		magnitude		characteristic function		attenuation dB		gain dB
δ_p	=	$1-M_p$	=	$\frac{2\delta_1}{1+\delta_1}$	=	$\frac{-1+\sqrt{1+K_p^2}}{\sqrt{1+K_p^2}}$	=	$1-10^{-A_p/20}$	=	$1-10^{G_p/20}$
$1-\delta_p$	=	M_p	=	$\frac{1-\delta_1}{1+\delta_1}$	=	$\frac{1}{\sqrt{1+K_p^2}}$	=	$10^{-A_p/20}$	=	$10^{G_p/20}$
$\frac{\delta_p}{2-\delta_p}$	=	$\frac{1-M_p}{1+M_p}$	=	δ_1	=	$\frac{-1+\sqrt{1+K_p^2}}{1+\sqrt{1+K_p^2}}$	=	$\frac{1-10^{-A_p/20}}{1+10^{-A_p/20}}$	=	$\frac{1-10^{G_p/20}}{1+10^{G_p/20}}$
$\frac{\sqrt{\delta_p(2-\delta_p)}}{1-\delta_p}$	=	$\frac{\sqrt{1-M_p^2}}{M_p}$	=	$\frac{2\sqrt{\delta_1}}{1-\delta_1}$	=	K_p	=	$\frac{\sqrt{1-10^{-A_p/10}}}{10^{-A_p/20}}$	=	$\frac{\sqrt{1-10^{G_p/10}}}{10^{G_p/20}}$
$-20\log_{10}(1-\delta_p)$	=	$-20\log_{10} M_p$	=	$20\log_{10}\frac{1+\delta_1}{1-\delta_1}$	=	$10\log_{10}(1+K_p^2)$	=	A_p	=	$-G_p$
$20\log_{10}(1-\delta_p)$	=	$20\log_{10} M_p$	=	$20\log_{10}\frac{1-\delta_1}{1+\delta_1}$	=	$-10\log_{10}(1+K_p^2)$	=	$-A_p$	=	G_p
δ_s	=	M_s	=	$\frac{\delta_2}{1+\delta_1}$	=	$\frac{1}{\sqrt{1+K_s^2}}$	=	$10^{-A_s/20}$	=	$10^{G_s/20}$
$\frac{2\delta_s}{2-\delta_p}$	=	$\frac{2M_s}{1+M_p}$	=	δ_2	=	$2\frac{\sqrt{\frac{1+K_p^2}{1+K_s^2}}}{1+\sqrt{1+K_p^2}}$	=	$2\frac{10^{-A_s/20}}{1+10^{-A_p/20}}$	=	$2\frac{10^{G_p/20}}{1+10^{G_p/20}}$
$\frac{\sqrt{1-\delta_s^2}}{\delta_s}$	=	$\frac{\sqrt{1-M_s^2}}{M_s}$	=	$\frac{\sqrt{(1+\delta_1)^2-\delta_2^2}}{\delta_2}$	=	K_s	=	$\frac{\sqrt{1-10^{-A_s/10}}}{10^{-A_s/20}}$	=	$\frac{\sqrt{1-10^{G_s/10}}}{10^{G_s/20}}$
$-20\log_{10}(\delta_s)$	=	$-20\log_{10} M_s$	=	$-20\log_{10}\frac{\delta_2}{1+\delta_1}$	=	$10\log_{10}(1+K_s^2)$	=	A_s	=	$-G_s$
$20\log_{10}(\delta_s)$	=	$20\log_{10} M_s$	=	$20\log_{10}\frac{\delta_2}{1+\delta_1}$	=	$-10\log_{10}(1+K_s^2)$	=	$-A_s$	=	G_s
$\delta_p = 1-\frac{\sqrt{2}}{2}$		$M_p = \frac{\sqrt{2}}{2}$				$K_p = 1$		$A_p \approx 3$		$G_p \approx -3$
$\delta_s \approx 0.1$		$M_s \approx 0.1$				$K_s = 10$		$A_s \approx 20$		$G_s \approx -20$
$\delta_s \approx 0.01$		$M_s \approx 0.01$				$K_s = 10^2$		$A_s \approx 40$		$G_s \approx -40$
$\delta_s \approx 0.0001$		$M_s \approx 0.0001$				$K_s = 10^4$		$A_s \approx 80$		$G_s \approx -80$
$\delta_s \approx 0.00001$		$M_s \approx 0.00001$				$K_s = 10^5$		$A_s \approx 100$		$G_s \approx -100$
$\delta_p \approx 0.005$		$M_p \approx 0.995$		$\delta_1 \approx 0.0099$		$K_p = 1/10$		$A_p \approx 0.043$		$G_p \approx -0.043$
$\delta_p \approx 0.00005$		$M_p \approx 0.99995$		$\delta_1 \approx 0.00001$		$K_p = 1/100$		$A_p \approx 0.0004$		$G_p \approx -0.0004$

there are numerous closed form expressions and numerical procedures for generating approximation functions.

The magnitude response of a lowpass *Butterworth* filter is smooth and monotonically decreases with respect to frequency. It is maximally flat at $\omega = 0$.

The *Chebyshev* filter, sometimes called Chebyshev type I filter, gives the smallest magnitude error over the entire passband. The magnitude response of the stopband monotonically decreases with respect to frequency.

The Butterworth and Chebyshev type I filters are *allpole* filters. They have no magnitude zeros.

The magnitude response of the Chebyshev type II filter, called also *Inverse Chebyshev* filter, is smooth and monotonically decreases with respect to frequency in the passband. It is maximally flat at $\omega = 0$ like the Butterworth filter. This filter gives the smallest magnitude error over the entire stopband.

The elliptic function filter, sometimes called *elliptic*, Cauer or Darlington filter, gives the smallest magnitude error over the entire passband and stopband.

The *Bessel* filter is an allpole filter like Butterworth and Chebyshev type I filters. Its magnitude response is smooth and monotonically decreases with respect to frequency. The main characteristics of this filter are that group delay is maximally flat at $\omega = 0$ and the step response overshoot is low.

Other types of filters exist, and exhibit good properties of group delay or in time domain [2]. There are also transitional approximations extensively published in open literature.

In digital filter theory there are additional approaches for solving the approximation problem (least square error method, windows approach) [3].

Which approximation should the designer choose?

One approach is to analyze all known approximations and to use all known procedures for calculating the magnitude function. The examination of all known approximations has a very high computational cost. Such an approach can be time consuming, too.

One straightforward approach is to specify a desired magnitude response and to try to find, numerically, an approximation closest to the response [3]. We also must add some constraints that are required by the numerical optimizer; the purpose of the additional constraints is just to simplify the design procedure. The numerical approach gives only one solution and we do not known whether a better solution exists.

It is interesting to notice [3] that good approximations are obtained by specifying some constraints in the transition region, which we normally treat as a "don't care" region.

In this paper we focus on only *one* approximation, the elliptic approximation, next, we find the design space, i.e. the range of design parameters that satisfy the specification, and keep the design parameters as symbols.

IV. Design space

In this section we define the design space. First, we map the specification into a standard form; next, we identify the design parameters; finally, we calculate the limits of the design parameters.

In the previous section we have shown several ways of presenting required specifications. Any low-pass filter can be specified by a set of four quantities as follows:

$$S_\delta = \{F_p, F_s, \delta_p, \delta_s\} \tag{7}$$

$$S_M = \{F_p, F_s, M_p, M_s\} \tag{8}$$

$$S_r = \{F_p, F_s, \delta_1, \delta_2\} \tag{9}$$

$$S_K = \{F_p, F_s, K_p, K_s\} \tag{10}$$

$$S_A = \{F_p, F_s, A_p, A_s\} \tag{11}$$

$$S_G = \{F_p, F_s, G_p, G_s\} \tag{12}$$

The relations between them have been summarized in Table 1. The symbols F_p and F_s designate the passband edge frequency and the stopband edge frequency, respectively, in Hz.

It is more convenient to transform a given specification S into the specification S_K because it provides a clearer relationship between the design parameters and the specification. In fact, we have to find a characteristic function $K(\omega)$.

There is an infinite number of characteristic functions that fit S. We will consider the *elliptic-function approximation*, because it fulfills the requirements with the minimal transfer function order. The minimal order can lead to the most economical solution (the minimal number of components, the minimal number of multiplications). Also, it will be shown that some other classical approximations are special cases of the elliptic approximating function.

The *prototype elliptic approximation*, K_e, is an nth-order rational function in real variable x

$$K_e(x) = \epsilon |R(n, a, x)| \tag{13}$$

where R, referred to as the *rational elliptic function*, satisfies the conditions

$$0 \le |R(n, a, x)| \le 1, \qquad |x| \le 1 \tag{14}$$

and

$$L(n, a) \le |R(n, a, x)|, \qquad |x| \ge a \tag{15}$$

L is the minimal value of the magnitude of R for $|x| \ge a$ and can be calculated as

$$L(n, a) = |R(n, a, a)| \tag{16}$$

The normalized transition band $1 < x < a$ is defined by

$$1 < |R(n, a, x)| < L(n, a), \qquad 1 < |x| < a \tag{17}$$

The parameter a is called the *selectivity factor*.

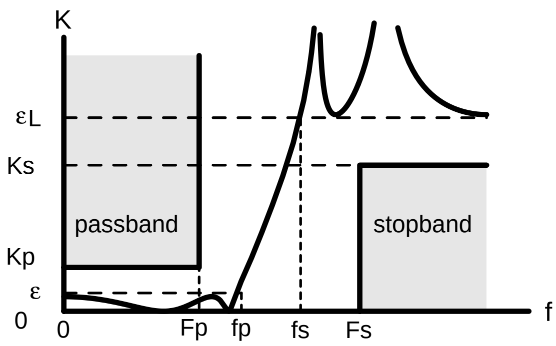

Fig. 8. Characteristic function.

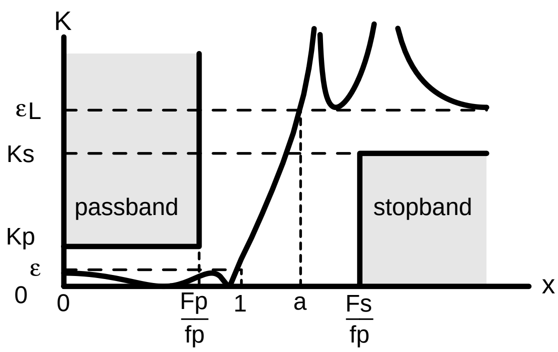

Fig. 9. Normalized characteristic function.

The parameter ϵ determines the maximal variation of K_e in the normalized passband $0 \le x \le 1$

$$0 \le K_e(x) \le \epsilon, \qquad |x| \le 1 \tag{18}$$

and is called the *ripple factor*.

The *elliptic approximation*, K, is a rational function in frequency ω rad/s, Figs. 8, 9,

$$K(\omega) = K_e(x), \qquad x = \frac{\omega}{2\pi f_p} \tag{19}$$

where f_p represents a design parameter that we call the *actual passband edge*. Traditionally, it has been set to $f_p = F_p$.

The four quantities, n, a, ϵ, and f_p, are collectively referred to as *design parameters* and can be expressed as a list of the form

$$D = \{n, a, \epsilon, f_p\} \tag{20}$$

Each of the listed parameters can take a value from a continuous (a,ϵ,f_p) or discrete (n) range of numbers. The order n is also referred to as the *filter order*.

It is known [3] that the ordinary elliptic function provides the minimal order, $n_{min} = n_{ellip}$, for a given specification. The maximal order, from the practical viewpoint, can be assumed to be $n_{max} = 2n_{min}$.

The selectivity factor, a, falls within the limits which are found by solving the equations [4]

$$R(n, a, a) = \frac{K_s}{K_p} \rightarrow a_{min} = a_{min}(n) \tag{21}$$

$$R\left(n, a, \frac{F_s}{F_p}\right) = \frac{K_s}{K_p}, \quad a > \frac{F_s}{F_p} \rightarrow a_{max} = a_{max}(n) \tag{22}$$

It follows from Eqs. (13) and (14) that the maximal value of ϵ must be equal to or less than K_p

$$\epsilon \leq K_p \tag{23}$$

The ripple factor quantifies the output signal amplitude, Y_m, with respect to the input signal amplitude, X_m. When $K(\omega) = 0$, both amplitudes have the same values, i.e.

$$Y_m = X_m, \qquad \text{for} \qquad K(\omega) = 0 \tag{24}$$

with $K(\omega) = \epsilon$ the amplitudes are different:

$$Y_m = \frac{X_m}{\sqrt{1+\epsilon^2}}, \qquad \text{for} \qquad K(\omega) = \epsilon \tag{25}$$

From the previous equations it follows that the smaller ϵ, the smaller the difference between the input and output amplitude.

What is the lower limit of ϵ?

The ripple factor, for $x > a$, must meet another condition $K_e(x) \geq K_s$, Fig. 9, thus

$$\epsilon L(n, a) \geq K_s \tag{26}$$

Therefore, the maximal and minimal values of ϵ has to be determined

$$\epsilon_{min} \leq \epsilon \leq \epsilon_{max} \tag{27}$$

From Eq. (23) we find the upper bound

$$\epsilon_{max} = K_p \tag{28}$$

From Eq. (26) we determine the lower bound

$$\epsilon_{min} = \frac{K_s}{L(n, a)} = \epsilon_{min}(n, a) \tag{29}$$

The maximal value of ϵ directly follows from specification, while the minimal value of ϵ depends on the order n and the selectivity factor a.

The actual passband edge, f_p, can take a value from the interval

$$f_{p,min} = \frac{F_s}{a_{max}} \leq f_p \leq \frac{F_s}{a_{min}} = f_{p,max} \tag{30}$$

Obviously, $f_{p,min} = f_{p,min}(n)$, and $f_{p,max} = f_{p,max}(n)$.

The set of all quartets $D = \{n, a, \epsilon, f_p\}$, satisfying the constraints $\{n_{min} \leq n \leq n_{max},\ a_{min} \leq a \leq a_{max},\ \epsilon_{min} \leq \epsilon \leq \epsilon_{max},\ f_{p,min} \leq f_p \leq f_{p,max}\}$, we call the *design space*.

$$D_S = \{D_{S,n}\}_{n=n_{min}, n_{min}+1, \ldots, n_{max}} \tag{31}$$

$$D_{S,n} = \left\{ \begin{array}{c} n \\ a_{min}(n) \leq a \leq a_{max}(n) \\ \epsilon_{min}(n, a) \leq \epsilon \leq \epsilon_{max} \\ f_{p,min}(n) \leq f_p \leq f_{p,max}(n)\} \end{array} \right. \tag{32}$$

The order n is an integer, it takes only the discrete numeric values, so, it is more convenient to express the design space, D_S, as a list of subspaces, $D_{S,n}$.

$$\left\{ \begin{array}{l} \left\{ \begin{array}{ccccc} & & n = n_{min} & & \\ a_{min}(n) & \leq & a & \leq & a_{max}(n) \\ \epsilon_{min}(n, a) & \leq & \epsilon & \leq & \epsilon_{max} \\ f_{p,min}(n) & \leq & f_p & \leq & f_{p,max}(n) \end{array} \right. \\ \left\{ \begin{array}{ccccc} & & n = n_{min}+1 & & \\ a_{min}(n) & \leq & a & \leq & a_{max}(n) \\ \epsilon_{min}(n, a) & \leq & \epsilon & \leq & \epsilon_{max} \\ f_{p,min}(n) & \leq & f_p & \leq & f_{p,max}(n) \end{array} \right. \\ \ldots \\ \left\{ \begin{array}{ccccc} & & n = n_{max} & & \\ a_{min}(n) & \leq & a & \leq & a_{max}(n) \\ \epsilon_{min}(n, a) & \leq & \epsilon & \leq & \epsilon_{max} \\ f_{p,min}(n) & \leq & f_p & \leq & f_{p,max}(n) \end{array} \right. \end{array} \right. \tag{33}$$

where

$$\begin{array}{ccccc} \epsilon_{min}(n) & > & \epsilon_{min}(n+1) & > & 0 \\ a_{min}(n) & > & a_{min}(n+1) & > & 1 \\ a_{max}(n) & < & a_{max}(n+1) & < & +\infty \end{array} \tag{34}$$

V. Basic design alternatives

This section lists a comprehensive set of design alternatives based on the design space. It is understood that the rational elliptic function can be readily constructed for a given set of design parameters [5], [6]. The advantages of the various designs are discussed.

Usually the designer selects the minimal order $n = n_{min}$. The design alternatives that follow are general and valid for any n from the design space. We assume that a specification, S, has been mapped to the form S_K.

A. *Design* D1

This design sets the three design parameters, $a = F_s/F_p$, $\epsilon = K_p$, $f_p = F_p$, directly from the specification, Fig. 10.

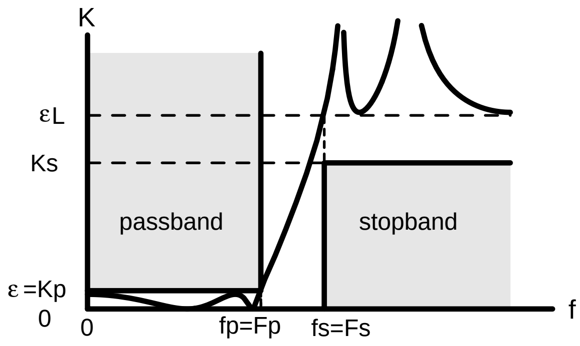

Fig. 10. Design D1.

The design D1 has higher attenuation in the stopband than it is required by S_A. We will choose this design when we prefer to achieve as large attenuation as possible in the stopband, i.e. $\epsilon L > K_s$.

B. *Design* D2

This design sets the two design parameters, $a = F_s/F_p$, $f_p = F_p$, directly from the specification, Fig. 11. The ripple factor is computed from $\epsilon = K_s/L(n, a)$.

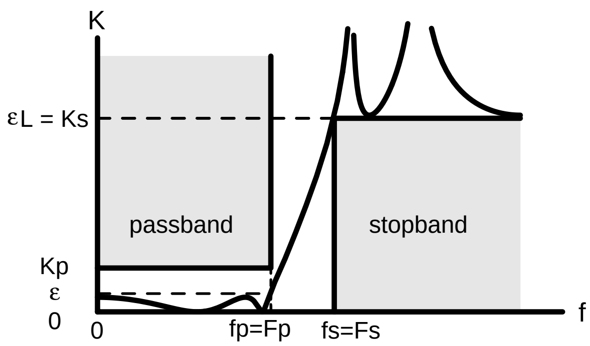

Fig. 11. Design D2.

The design D2 has lower attenuation in the passband than it is required by S_A. We will choose this design when we prefer to achieve as low attenuation as possible in the passband, i.e. $\epsilon < K_p$. Also, this design is suitable when filter element imperfections can significantly change the magnitude in the passband; in that case, we achieve the highest attenuation margin in the passband (the margin is $K_p - \epsilon$, Fig. 11), and we expect that the imperfections of the implemented filter will not violate the specification.

C. *Design* D3a

In this design we choose the minimal selectivity factor, $a = a_{min}$, and set the two design parameters, $\epsilon = K_p$, $f_p = F_p$, directly from the specification, Fig. 12.

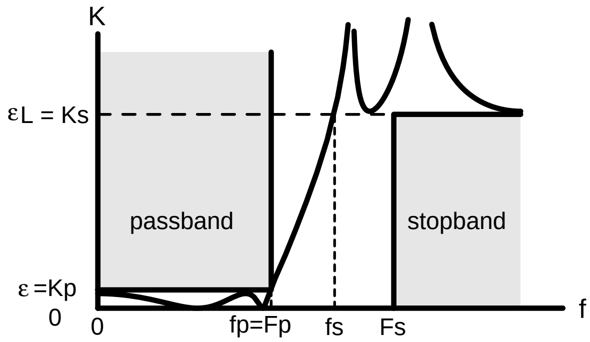

Fig. 12. Design D3a.

The design D3a has the sharpest magnitude response. When undesired signals exist in the transition region we prefer design D3a, because it rejects the undesired signals as much as possible.

Disadvantages of D3a can be very high Q-factors and large variation of the group-delay in the passband.

D. *Design* D3b

For this design we choose the minimal selectivity factor, $a = a_{min}$, (the same as in Design 3a) and set the ripple factor, $\epsilon = K_p$, directly from the specification, Fig. 13. The actual passband edge is computed from $f_p = f_{p,max} = F_s/a$.

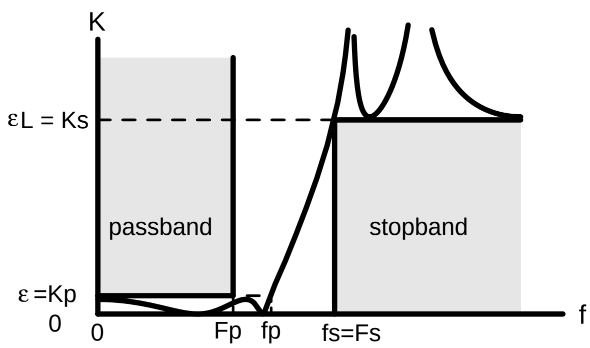

Fig. 13. Design D3b.

The design D3b has the sharpest magnitude response (the same as D3a). When the desired signals exist in the transition region we prefer the design D3b, because it attenuates the desired signals as low as possible.

A disadvantage of the design D3b can be very high Q-factors. Although the variation of the group-delay can be high, its maximal value can be moved into the transition region, so, the group-delay variation can be acceptable in the passband.

E. *Design* D4a

In this design we choose the maximal selectivity factor, $a = a_{max}$, and set the two design parameters, $\epsilon = K_p$, $f_p = F_p$, directly from the specification, Fig. 14.

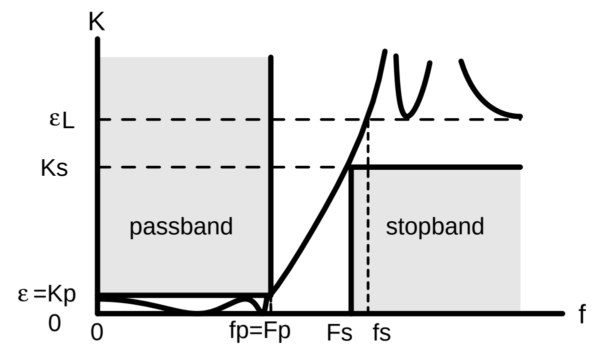

Fig. 14. Design D4a.

The design D4a (like the design D1) has higher attenuation in the stopband than it is required by S_A, except at the stopband edge frequency. We choose this design when we prefer to achieve as large attenuation as possible in the stopband, i.e. $\epsilon L > K_s$, except at $f = F_s$.

The design D4a has a smoother magnitude response, and that is the main reason for lower Q-factors and smaller variation of the group-delay in the passband.

F. Design D4b

For this design we choose the maximal selectivity factor, $a = a_{max}$, (the same as in the design D4a), Fig. 15, and calculate the ripple factor from $\epsilon = K_s/L(n, a)$. The actual passband edge is computed from $f_p = f_{p,min} = F_s/a$.

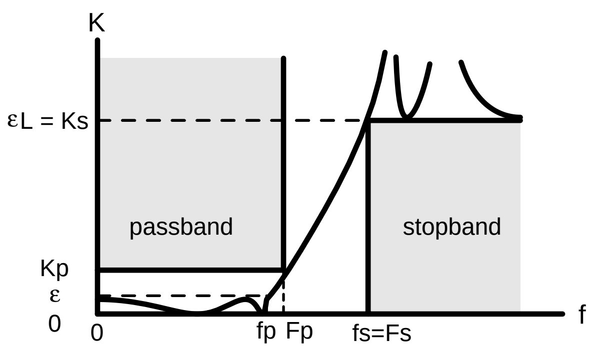

Fig. 15. Design D4b.

The design D4b (like the design D2) has lower attenuation in the passband than it is required by S_A, except at the passband edge frequency.

We choose this design when we prefer to achieve as low attenuation as possible in the passband, i.e. $\epsilon < K_p$, except at $f = F_p$.

The design D4b has a smoother magnitude response. This design usually yields very low Q-factors and small variation of the group-delay in the passband. The design D4b has the lowest ripple factor ϵ.

It should be noticed that there exists a straightforward procedure for computing the ripple factor, ϵ, for a given selectivity factor a, that yields the minimal Q-factors [7]. We call this design D5.

A disadvantage of D4b (also, D3a, D3b, D4a) is lack of any attenuation margin. Any imperfection, usually in implementation step (like element tolerances or coefficient quantization), can violate the specification.

G. Remarks on the design alternatives

The approach that we propose in the previous sections we have programmed in *Mathematica* [10]. Several design examples are exercised for an illustrative specification: $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$.

First, $S_A = \{3,\ 3.225,\ 0.2,\ 40\}$ is transformed into $S_K = \{3, 3.225, 0.2171, 100\}$. Next, all designs are calculated, and the attenuations are plotted, Figs. 16 - 31. The design parameters, the actual stopband edge, the maximal attenuation in the passband, and the minimal attenuation in the stopband, are summarized in Tables 2 and 3. The maximal Q-factor is presented in the last column.

If technological requirements impose a maximal value of Q-factors, e.g. $Q_{max} = 20$ for active RC filters, Table 1 reveals than all six design alternatives fail. The design D4b is the best suboptimal solution.

An advanced design technique, the design D5 [7] with doubled poles [12], achieves the maximum Q-factor lower than 20. This is paid by increasing the filter order; the actual filter order is 12, that is much higher than the minimal order $n = 8$. Although the filter order has been increased, the implementation can be more cost effective [8] [9]; the lower tolerance components can be used and the magnitude response of the implemented filter satisfies the specifications [8].

We can enlarge the design space by increasing the filter order from $n = 8$ to $n = 9$. The corresponding attenuations are plotted on Figs. 26 - 31 and the design results are summarized in Table 3. The Q-factor of the design D4b is 20.9 that is very close to the required maximal value ($Q = 20$).

In practice, we choose the most suitable design, D, from the determined design space D_S. Thus, we can try to meet various technological requirements (maximal Q-factors, maximal element tolerances, prescribed values of coefficients of digital IIR filter [11] [13]), and advanced specifications (maximal group delay variation, maximal rise time, maximal overshoot in step response, maximal settling time).

It should be noticed that the design parameters of the design D5 belong to the design space D_S; D5 and its modification D5a [7] yield, also, elliptic function filters. By further increasing the filter order, we arrive at the Chebyshev type approximation, for $n = n_C$; the design D4a degenerates from rational into polynomial (allpole) filter. Alternatively, for the same order, $n = n_C$, the design D4b yields an Inverse Chebyshev type filter. When the filter order is equal to the order of the Butterworth type filter, the design D5 transforms into an allpole Butterworth type filter. This means that the classical filter types, Chebyshev, Inverse Chebyshev and Butterworth, are just special cases of the elliptic function filters, and are contained within the design space D_S.

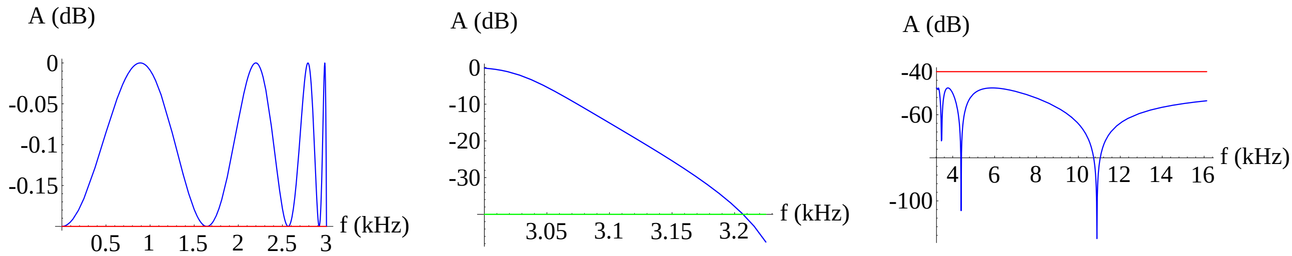

Fig. 16. Design 1: $D = \{n_{min}, F_s/F_p, \epsilon_{max}, F_p\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

Table 2. Design summary for $n = n_{min}$.

Design	n	a	ϵ	f_p (kHz)	f_s (kHz)	a_p (dB)	a_s (dB)	Q_{max}
D 1	8	1.075	0.2171	3.	3.225	0.2	47.55	29.92
D 2	8	1.075	0.09097	3.	3.225	0.03579	40.	24.24
D 3a	8	1.043	0.2171	3.	3.129	0.2	40.	42.12
D 3b	8	1.043	0.2171	3.092	3.225	0.2	40.	42.12
D 4a	8	1.083	0.2171	3.	3.25	0.2	49.14	28.18
D 4b	8	1.083	0.07579	2.977	3.225	0.02487	40.	22.07
D 5	11	1.075	0.006257	3.	3.225	0.00017	44.07	31.36
D 5a	10	1.079	0.01	2.989	3.225	0.0004343	40.	27.27
D 5	$2 \times 6 = 12$	1.075	0.08607	3.	3.225	0.06411	42.67	15.95

Table 3. Design summary for $n = n_{min} + 1$.

Design	n	a	ϵ	f_p (kHz)	f_s (kHz)	a_p (dB)	a_s (dB)	Q_{max}
D 1	9	1.075	0.2171	3.	3.225	0.2	56.66	37.36
D 2	9	1.075	0.03188	3.	3.225	0.004412	40.	25.76
D 3a	9	1.022	0.2171	3.	3.066	0.2	40.	81.2
D 3b	9	1.022	0.2171	3.156	3.225	0.2	40.	81.2
D 4a	9	1.098	0.2171	3.	3.294	0.2	61.4	32.14
D 4b	9	1.098	0.01847	2.937	3.225	0.001481	40.	20.87
D 5	12	1.075	0.006257	3.	3.225	0.00017	44.07	34.67
D 5a	10	1.079	0.01	2.989	3.225	0.0004343	40.	27.27
D 5	$2 \times 6 = 12$	1.075	0.08607	3.	3.225	0.06411	42.67	15.95

VI. Conclusion

Opposite to the conventional approaches, that return only one design and hide a wealth of alternative filter designs, the advanced design techniques that we present in this paper find a comprehensive set of optimal designs to represent the infinite solution space.

The primary benefit of this paper is convenient access to the latest advances in algorithms for analog and digital IIR filter design. These advanced techniques can design many types of filters that conventional techniques cannot design. A secondary benefit is a collection of case studies for filter designs that are suitable for non experienced designers.

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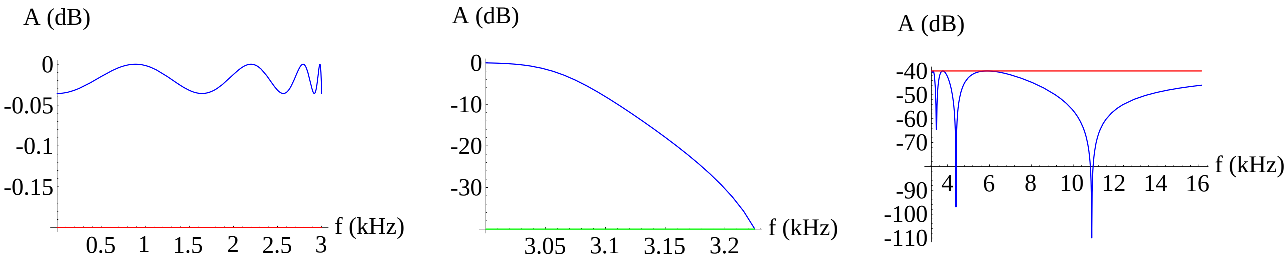

Fig. 17. Design 2: $D = \{n_{min}, F_s/F_p, \epsilon < \epsilon_{max}, F_p\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

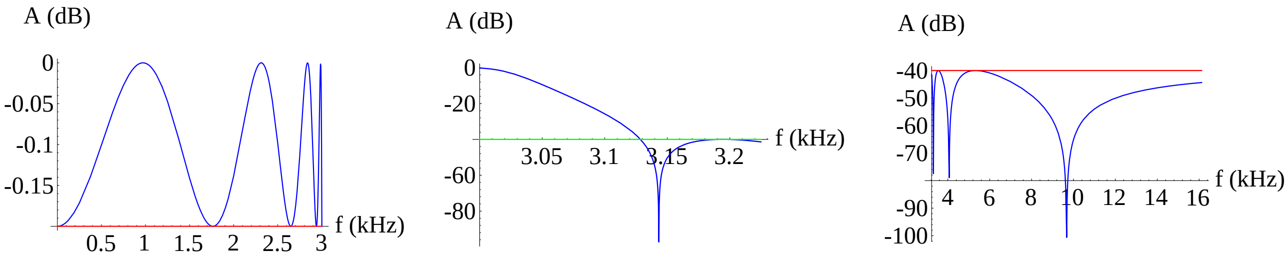

Fig. 18. Design 3a: $D = \{n_{min}, a_{min}, \epsilon_{max}, F_p\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

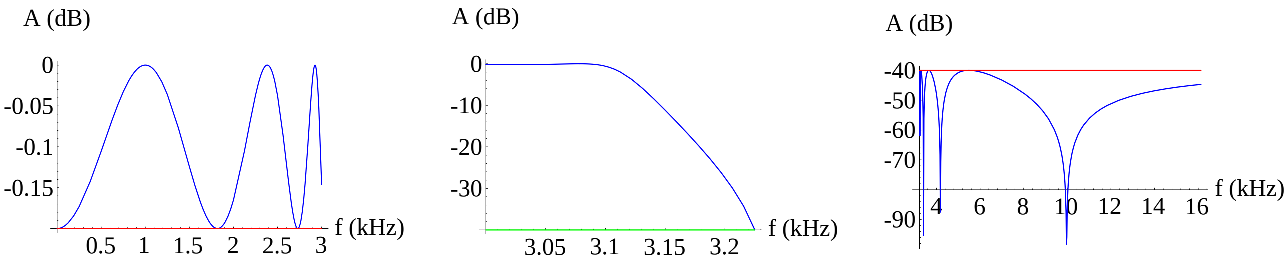

Fig. 19. Design 3b: $D = \{n_{min}, a_{min}, \epsilon_{max}, F_s/a, \}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

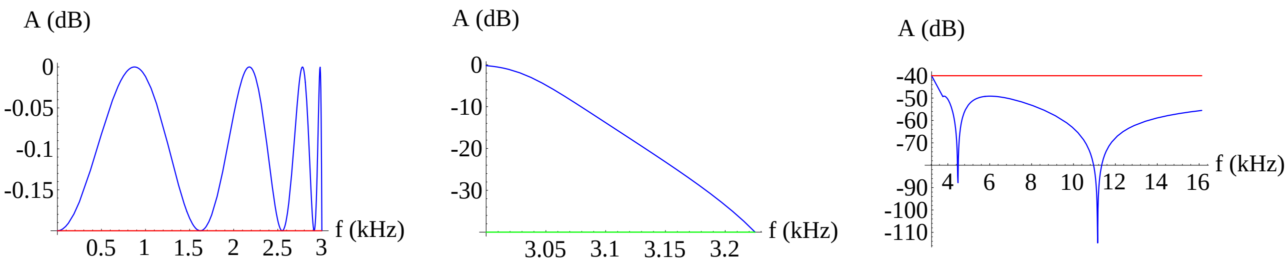

Fig. 20. Design 4a: $D = \{n_{min}, a_{max}, \epsilon_{max}, F_p, \}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

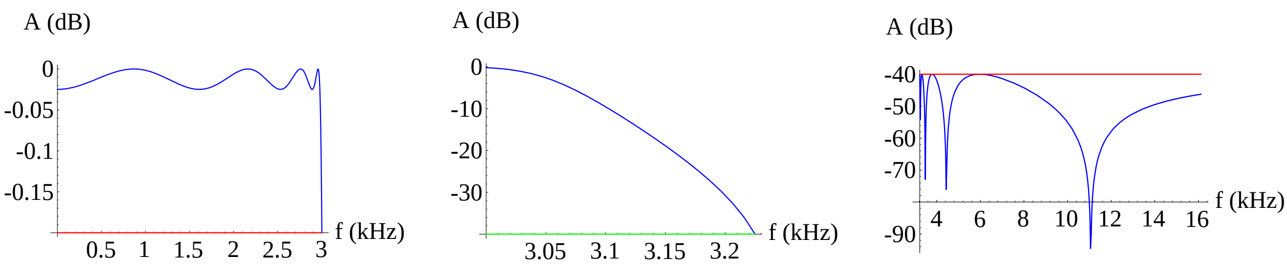

Fig. 21. Design 4b: $D = \{n_{min}, a_{max}, \epsilon_{min}, F_s/a, \}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

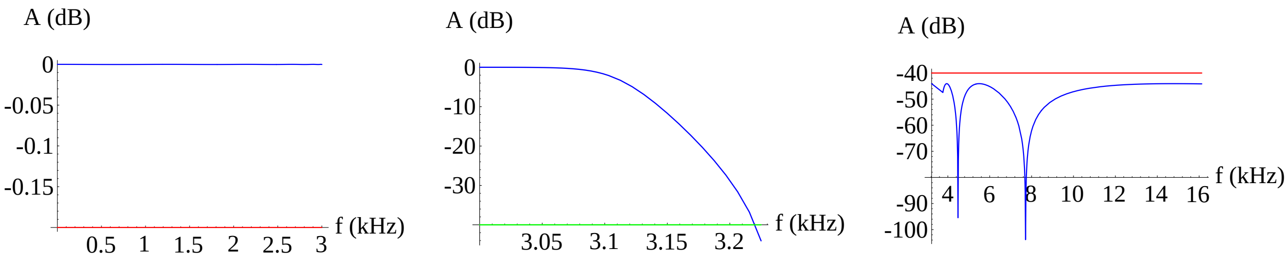

Fig. 22. Design 5: $D = \{n > n_{min}, F_s/F_p, \epsilon << \epsilon_{max}, F_p\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

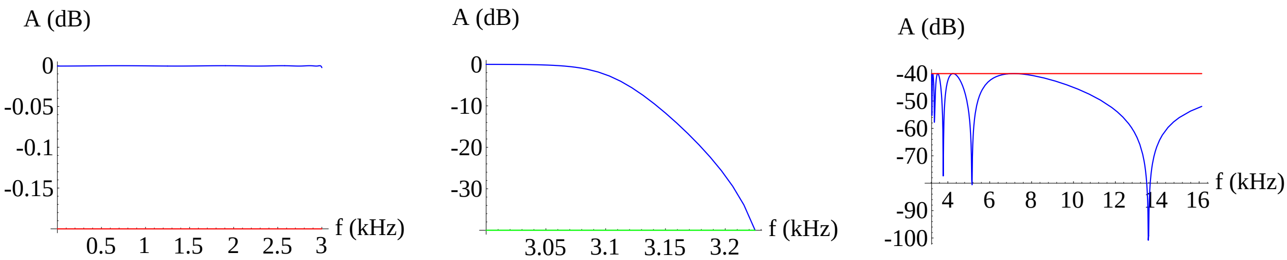

Fig. 23. Design 5a: $D = \{n > n_{min}, a > F_s/F_p, \epsilon << \epsilon_{max}, F_s/a\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

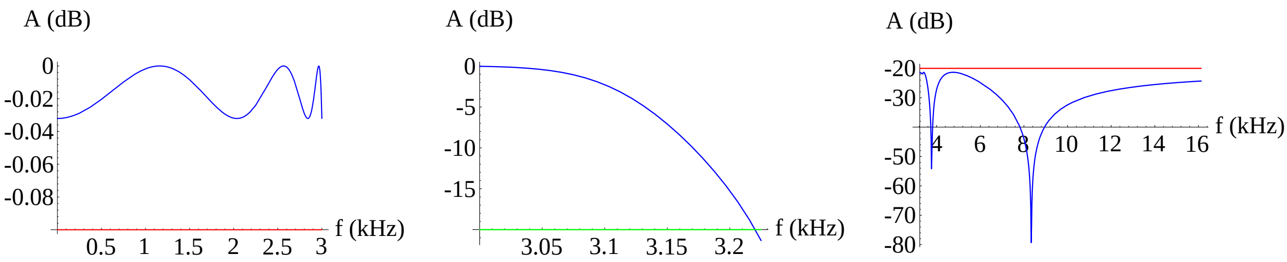

Fig. 24. Design 5: $D = \{n > n_{min}, F_s/F_p, \epsilon_{max}, F_p\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.1\text{dB}, A_s = 20\text{dB}\}$

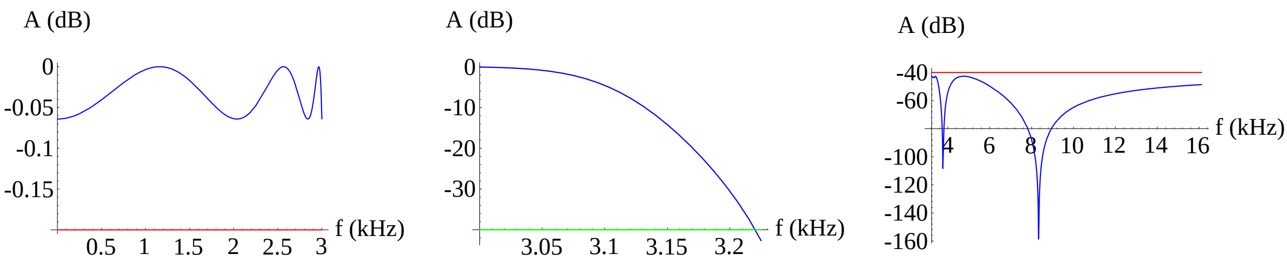

Fig. 25. Design $D = \{2 \times 5, F_s/F_p, \epsilon_{max}, F_p\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

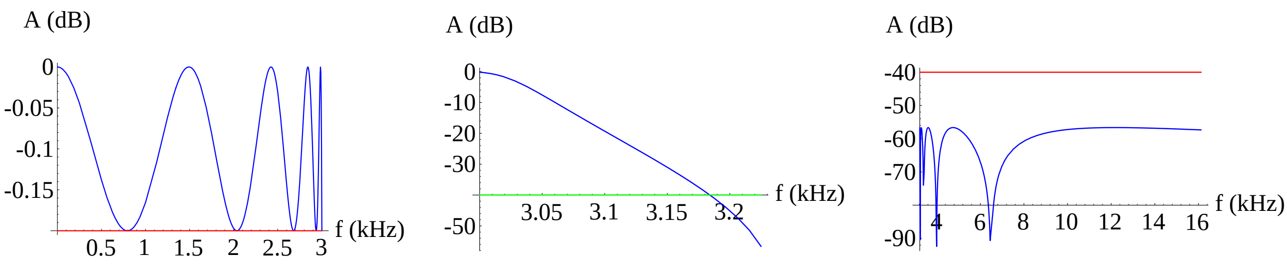

Fig. 26. Design 1: $D = \{n_{min} + 1, F_s/F_p, \epsilon_{max}, F_p\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

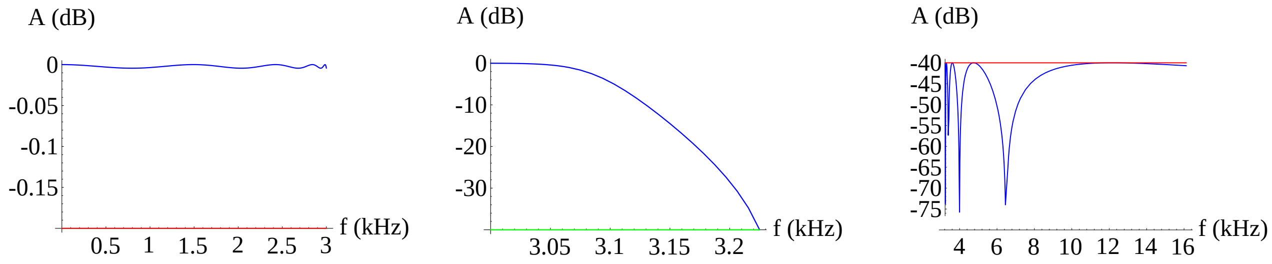

Fig. 27. Design 2: $D = \{n_{min} + 1, F_s/F_p, \epsilon < \epsilon_{max}, F_p\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

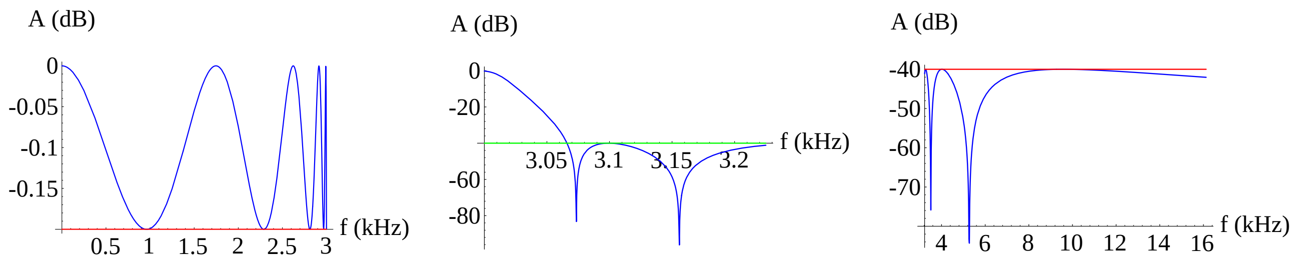

Fig. 28. Design 3a: $D = \{n_{min} + 1, a_{min}, \epsilon_{max}, F_p\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

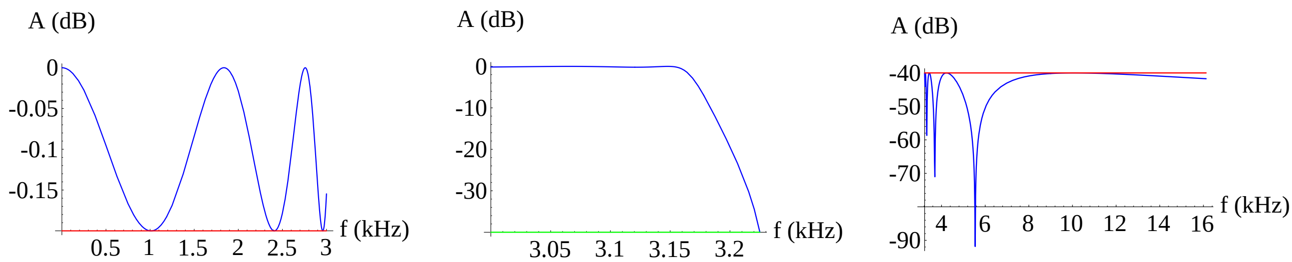

Fig. 29. Design 3b: $D = \{n_{min} + 1, a_{min}, \epsilon_{max}, F_s/a\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

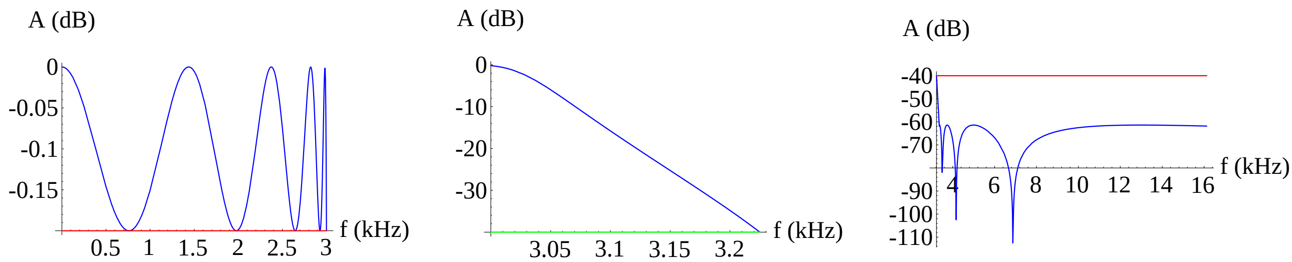

Fig. 30. Design 4a: $D = \{n_{min} + 1, a_{max}, \epsilon_{max}, F_p\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$

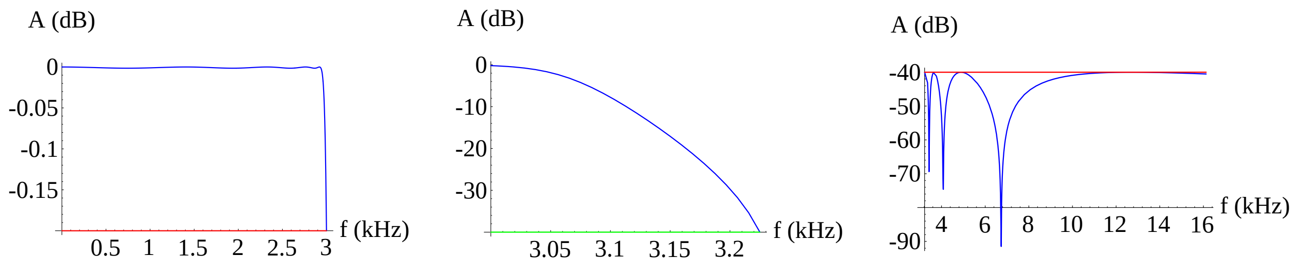

Fig. 31. Design 4b: $D = \{n_{min} + 1, a_{max}, \epsilon_{min}, F_s/a\}$ for $S_A = \{F_p = 3\text{kHz}, F_s = 3.225\text{kHz}, A_p = 0.2\text{dB}, A_s = 40\text{dB}\}$