Goal

- Design a new family of wavelets: *Biorthogonal Quincunx Coifman Wavelets*
  - 2-D nonseparable
  - compactly supported
  - biorthogonal
  - quincunx down- and up-sampling

- Useful properties:
  - zero-phase filterbanks
  - dyadic rational filter coefficients
  - cardinal scaling functions
  - converge to ideal filters asymptotically
  - closed-form formulae

- Promising applications:
  - image compression
  - sampling/interpolation
Motivation

- Why biorthogonal wavelets?
  - more flexible design
  - linear-phase filterbanks
  - cardinal scaling functions

- Why quincunx wavelets?
  - finer multiresolution analysis
  - allow nonseparable sampling
  - allow nonseparable filtering
  - more isotropic bases
  - better adaption to human visual system

- Why Coifman wavelets?
  - tradeoff between two types of vanishing moments
  - easy to achieve linear phase
  - useful in sampling/interpolation
1-D Biorthogonal Coiflets

- Synthesis filter of order \( l \):

\[
H_l(\omega) = \left( \frac{1 + \cos \omega}{2} \right)^k \\
\times \sum_{p=0}^{k-1} \binom{k-1}{p} \left( \frac{1 - \cos \omega}{2} \right)^p
\]

where \( l = 2k \)

- Analysis filter of order \((l, l')\):

\[
\tilde{H}_{l,l'}(\omega) = 2H_{l'}(\omega) + H_l(\omega) - 2H_{l'}(\omega)H_l(\omega)
\]

- Useful properties:
  - zero-phase filterbanks
  - dyadic rational filter coefficients
  - converges to ideal filters asymptotically
  - cardinal scaling functions
  - excellent for data compression
Quincunx Filterbank and Wavelet

Iterated Analysis Filterbank

Iterated Synthesis Filterbank

\[ D = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ or } D = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \]
Three Equivalent Definitions

- All moments up to order \((l-1)\) of the scaling function \(\phi(t)\) and the wavelet \(\psi(t)\) vanish:

\[
\int_{\mathbb{R}^2} t^k \phi(t) \, dt = \delta[k], \quad \int_{\mathbb{R}^2} t^k \psi(t) \, dt = 0
\]

for \(k = [k_1, k_2] \in \mathbb{Z}^2, 0 \leq k_1 \leq l - 1, 0 \leq k_2 \leq l - 1\), and \(k_1 + k_2 \leq l - 1\), where \(\delta[k]\) denotes the Kronecker delta symbol and \(t^k\) denotes \(t_1^{k_1} t_2^{k_2}\).

- All moments up to order \((l-1)\) of the lowpass and highpass filters vanish:

\[
\sum_{n \in \mathbb{Z}^2} n^k h_l[n] = \delta[k], \quad \sum_{n \in \mathbb{Z}^2} n^k g_l[n] = 0
\]

for \(k, k_1,\) and \(k_2\) as above, where \(n^k\) denotes \(n_1^{k_1} n_2^{k_2}\).

- The frequency response of the lowpass filter has a zero of order \(l\) at the origin and the aliasing frequency \(\pi = [\pi, \pi]\):

\[
\left. \frac{\partial^{k_1+k_2} H_l(\omega_1, \omega_2)}{\partial \omega_1^{k_1} \partial \omega_2^{k_2}} \right|_{\omega=0, \pi} = 0
\]

for \(k_1\) and \(k_2\) as above.
Why Vanishing Moments?

- Vanishing moments of wavelet
  - Combination of shifted scaling functions can approximate smooth functions accurately.
  - The wavelet coefficients of a smooth function decay rapidly.
  - Necessary condition for a wavelet to be smooth.
  - Correspond to zeros of $H_l(\omega)$ at $\pi$.

- Vanishing moments of scaling function
  - The uniform samples of a smooth function can approximate its wavelet expansion coefficients accurately.
  - Imposing zero moments on a scaling function improves its symmetry and makes the filterbank close to linear-phase.
  - Correspond to zeros of $H_l(\omega)$ at $0$. 
Filterbank Design

- McClellan transformation-based design:
  - 1-D prototype filter:
    \[
    H_l(\omega) = \frac{1}{2} + \sum_{k=1}^{l/2} 2h_l[2k - 1] T_{2k-1}[\cos \omega]
    \]
    where \( T_p[.] \) denotes the \( p \)th-order Chebyshev polynomial
  - transformation function:
    \[
    F(\omega) = \frac{1}{2}(\cos \omega_1 + \cos \omega_2)
    \]
  - transformed 2-D filter:
    \[
    H_l(\omega) = \frac{1}{2} + \sum_{k=1}^{l/2} 2h_l[2k - 1] T_{2k-1}[F(\omega)]
    \]
- Advantages:
  - simple
  - preserve the properties of 1-D filter
Sampling and Interpolation

- The synthesis scaling function in a BQCW system of any order is \textit{cardinal}; i.e., for any \( n \in \mathbb{Z}^2 \),

\[
\phi(n) = \delta[n]
\]

which is a useful property in sampling and interpolation.

- The approximation

\[
\tilde{f}(t) = \sum_{k \in \mathbb{Z}^2} f(D^{-i}k) \phi(D^i t - k)
\]

is exact at the quincunx sampling grid:

\[
\tilde{f}(D^{-i}n) = f(D^{-i}n)
\]

for any \( n \in \mathbb{Z}^2 \) and any integer \( i \).

- Wavelet expansion coefficients can be accurately approximated by function samples due to the vanishing moment conditions on the scaling function:

\[
\langle f(t), 2^{i/2} \phi(D^i t - k) \rangle \approx f(D^{-i} k).
\]
Asymptotic Convergence

- The frequency responses of BQCW filters converge pointwise to the *ideal diamond-shaped halfband lowpass filters* as their orders tend to infinity:

\[
\lim_{l \to \infty} H_l(\omega) = \begin{cases} 
1 & \text{if } |\omega_1| + |\omega_2| < \pi \\
1/2 & \text{if } |\omega_1| + |\omega_2| = \pi \\
0 & \text{otherwise}
\end{cases}
\]

\[
\lim_{l, l' \to \infty} \tilde{H}_{l, l'}(\omega) = \begin{cases} 
1 & \text{if } |\omega_1| + |\omega_2| \leq \pi \\
0 & \text{otherwise}
\end{cases}
\]

- The convergence of \( H_l(\omega) \) is monotonic:

\[
H_l(\omega) \leq H_{l+1}(\omega) \quad \text{if } |\omega_1| + |\omega_2| \leq \pi
\]

\[
H_l(\omega) \geq H_{l+1}(\omega) \quad \text{if } |\omega_1| + |\omega_2| > \pi
\]

- The convergence of \( H_l(\omega) \) does not exhibit any Gibbs-like phenomenon.

- The convergence of \( \tilde{H}_{l, l'}(\omega) \) exhibits a one-sided Gibbs-like phenomenon.
A Design Example

Frequency Response of Synthesis Filter

Frequency Response of Analysis Filter
Energy Compaction

BSGAM−7/9
BQCW−7/9

PERCENTAGE OF THRESHOLDED COEFFICIENTS (%)

PSNR

50 60 70 80 90

Lena

PERCENTAGE OF THRESHOLDED COEFFICIENTS (%)

PSNR

50 60 70 80 90

Peppers

PERCENTAGE OF THRESHOLDED COEFFICIENTS (%)

PSNR

50 60 70 80 90

Barbara

PERCENTAGE OF THRESHOLDED COEFFICIENTS (%)

PSNR

50 60 70 80 90

Goldhill