JOINT OPTIMIZATION OF MULTIPLE BEHAVIORAL AND IMPLEMENTATION PROPERTIES OF ANALOG FILTER DESIGNS

*Niranjan Damera-Venkata and Brian L. Evans**

Dept. of Electrical and Computer Eng.
Engineering Science Building
The University of Texas at Austin
Austin, TX 78712-1084 USA
damera-v@ece.utexas.edu
bevans@ece.utexas.edu

Miroslav D. Lutovac and Dejan V. Tošić

Faculty of Electrical Engineering
Bulevar Revolucije 73
University of Belgrade
11000 Belgrade, Yugoslavia
lutovac@iritel.bg.ac.yu
tosic@telekom.etf.bg.ac.yu

ABSTRACT

This paper presents an extensible framework for optimizing analog filter designs for multiple behavioral and implementation properties. We demonstrate the framework using the behavioral properties of magnitude response, phase response, and peak overshoot, and the implementation property of quality factors. We represent the analog filter in terms of its poles and zeroes. We match the constrained non-linear optimization problem to a sequential quadratic programming (SQP) problem, and develop symbolic mathematical software to translate the SQP formulation into working MATLAB programs to optimize analog filter designs. The automated approach avoids errors in algebraic calculations and errors in transcribing the mathematical equations in software. The packages are freely distributable.

1. INTRODUCTION

Classical analog filter design techniques optimize one filter property subject to constraints on the magnitude response. In designing and implementing analog filters, several behavioral properties (e.g. magnitude response, phase response and peak overshoot) and implementation properties (e.g. quality factors) may be important. For example, anti-aliasing filters require a near linear phase response while meeting a set of magnitude specifications [1].

This paper presents a formal extensible framework for optimizing analog filter designs for multiple behavioral and implementation properties. We demonstrate the framework using the behavioral properties of magnitude response, phase response, and peak overshoot, and the implementation property of quality factors. The framework takes an existing analog filter design, e.g. one designed using a classical numeric approach or a modern symbolic approach [2], and jointly optimizes any combination of these four properties subject to constraints on these four properties. We match the constrained non-linear optimization problem to a sequential quadratic programming (SQP) problem and develop symbolic mathematical software to translate the SQP formulation into working MATLAB programs that can optimize analog filter designs. SQP requires that the objective function [3] and the constraints [4] be real-valued and twice continuously differentiable with respect to the free parameters. The free parameters are the pole and zero locations, which we use to represent the analog filter. SQP relies on the gradients of the objective function and constraints. SQP methods have been previously applied to optimizing loss and delay in digital filter designs [5] and optimizing even-order all-pole filter designs [6]. In this paper, we reformulate our results in [6] to include an even number of zeros in the analog filter to be optimized.

Using the symbolic mathematics environment Mathematica, we program the objective function and constraints, compute their gradients symbolically, and generate MATLAB code for the objective function and constraints as well as their gradients. The generated MATLAB code calls the SQP procedure `constr` in the Optimization Toolbox to solve the constrained non-linear optimization problem. In Mathematica, a designer can add, delete, and change cost measures and constraints for a given property, and our symbolic software will then regenerate the MATLAB numerical optimization code. We have bridged the gap between the symbolic work designers do on paper and the working computer implementation, thereby eliminating algebraic errors in hand calculations and bugs in coding the software implementation. Our software is available at http://www.ece.utexas.edu/~bevans/projects/syn_filter_software.html.

Section 2 reviews notation. Section 3 derives a family of weighted, differentiable objective functions to measure the deviation in magnitude response, deviation in linear phase response, quality factors, and peak overshoot of the step response, of an analog filter. In the derivation, we find a new analytic approximation for the peak overshoot. Section 4 converts filter specifications into differentiable constraints. Section 5 gives an example of an optimized filter design.

2. NOTATION

We represent an analog filter by its n complex conjugate pole pairs $p_k = a_k \pm jb_k$ where $a_k < 0$ and its r complex conjugate zero pairs $z_l = c_l \pm jd_l$ where $c_l < 0$, such that

*This research was supported by an NSF CAREER Award under Grant MIP-9702707.

$r \leq n$. The magnitude and unwrapped phase responses of an all-pole filter, expressed as real-valued differentiable functions, are

$$
\begin{aligned}
|G(j\omega)| &= \prod_{k=1}^{n} \frac{a_k^2 + b_k^2}{\sqrt{a_k^2 + (\omega + b_k)^2}\sqrt{a_k^2 + (\omega - b_k)^2}} \\
&= \prod_{k=1}^{n} \frac{a_k^2 + b_k^2}{\sqrt{(\omega^2 + 2(a_k^2 - b_k^2))\omega^2 + (a_k^2 + b_k^2)^2}}
\end{aligned} \tag{1}
$$

$$
\angle G(j\omega) = \sum_{k=1}^{n} \arctan\left(\frac{\omega - b_k}{a_k}\right) + \arctan\left(\frac{\omega + b_k}{a_k}\right) \tag{2}
$$

We factor the polynomial under the square root in (1) into Horner's form because it has better numerical properties. Together with the zero pairs, the magnitude and unwrapped phase responses are

$$
|H(j\omega)| = |G(j\omega)| \times \prod_{l=1}^{r} \frac{\sqrt{(\omega^2 + 2(c_l^2 - d_l^2))\omega^2 + (c_l^2 + d_l^2)^2}}{c_l^2 + d_l^2} \tag{3}
$$

$$
\angle H(j\omega) = \angle G(j\omega) - \sum_{l=1}^{r} \arctan\left(\frac{\omega - d_l}{c_l}\right) + \arctan\left(\frac{\omega + d_l}{c_l}\right) \tag{4}
$$

In this paper, Q represents quality factors, ϵ represents a small positive number, σ denotes deviation, m represents slope of a line, and t is time.

3. OBJECTIVE FUNCTIONS

In this section, we derive measures of closeness to an ideal magnitude and phase response, quality factors, and peak overshoot. The objective function is a non-negative function that it is weighted combination of these measures.

3.1. Deviation in the Magnitude Response

Based on the notation in Figure 1, the five components of the objective function relating to the deviation from an ideal magnitude response in the least squares sense are:

$$
\sigma_{sb1} = \int_{0}^{\omega_{s1}} F_{s1}(\omega) |H(j\omega)|^2 \, d\omega \tag{5}
$$

$$
\sigma_{tb1} = \int_{\omega_{s1}}^{\omega_{p1}} F_{t1}(\omega) \left(|H(j\omega)| - (m_1\, \omega - m_1\, \omega_{s1})\right)^2 \, d\omega \tag{6}
$$

$$
\sigma_{pb} = \int_{\omega_{p1}}^{\omega_{p2}} F_p(\omega) \left(|H(j\omega)| - 1\right)^2 \, d\omega \tag{7}
$$

$$
\sigma_{tb2} = \int_{\omega_{p2}}^{\omega_{s2}} F_{t2}(\omega) \left(|H(j\omega)| - (m_2\, \omega - m_2\, \omega_{s2})\right)^2 \, d\omega \tag{8}
$$

$$
\sigma_{sb2} = \int_{\omega_{s2}}^{\infty} F_{s2}(\omega) |H(j\omega)|^2 \, d\omega \tag{9}
$$

where $F_p(\omega)$, $F_{t1}(\omega)$, $F_{t2}(\omega)$, and $F_s(\omega)$ are integrable weighting functions, and m_1 and m_2 are the slopes of the ideal response in the transition regions defined as $m_1 = 1/(\omega_{p1} - \omega_{s1})$ and $m_2 = 1/(\omega_{p2} - \omega_{s2})$.

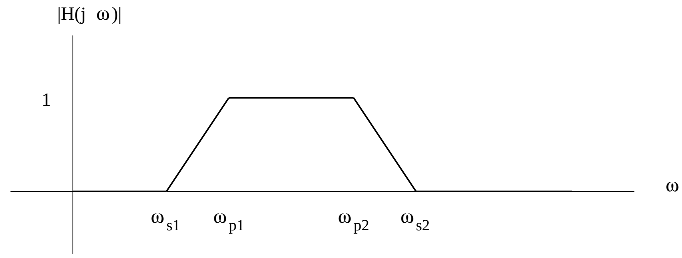

Figure 1: The ideal magnitude response

3.2. Deviation in the Phase Response

For the passband response, the objective function measures the deviation from linear phase over some range of frequencies (usually over the passband):

$$
\sigma_{phase} = \int_{\omega_1}^{\omega_2} \left(\angle H(j\omega) - m_{lp}\omega\right)^2 \, d\omega \tag{10}
$$

where m_{lp} is the ideal slope of the linear phase response. Unfortunately, one does not know the value of m_{lp} *à priori*. We can compute it as the slope of the line in ω that minimizes (10):

$$
\min_{m_{lp}} \int_{\omega_1}^{\omega_2} \left(\angle H(j\omega) - m_{lp}\omega\right)^2 \, d\omega \tag{11}
$$

In (11), the $H(j\omega)$ term does not depend on m_{lp}, so the integrand is quadratic in m_{lp}. To find the minimum, we take the derivative with respect to m_{lp}, set it to zero, and solve for m_{lp}:

$$
m_{lp} = \frac{\int_{\omega_1}^{\omega_2} \angle H(j\omega)\, \omega \, d\omega}{\int_{\omega_1}^{\omega_2} \omega^2 \, d\omega} \tag{12}
$$

After evaluating the integrals,

$$
m_{lp} = \frac{3}{2(\omega_2^3 - \omega_1^3)} \times \left[\sum_{k=1}^{n} \left[\, f_{lp1}(\omega_2) - f_{lp1}(\omega_1)\, \right] - \sum_{l=1}^{r} \left[\, f_{lp2}(\omega_2) - f_{lp2}(\omega_1)\, \right] \right] \tag{13}
$$

where $f_{lp1}(\omega)$ is

$$
f_{lp1}(\omega) = 2\omega a_k + (b_k^2 - a_k^2 - \omega^2) \times \left(\arctan\left(\frac{\omega - b_k}{a_k}\right) + \arctan\left(\frac{\omega + b_k}{a_k}\right)\right) + a_k b_k \left(\log\left(1 + \frac{(\omega - b_k)^2}{a_k^2}\right) - \log\left(1 + \frac{(\omega + b_k)^2}{a_k^2}\right)\right)
$$

and $f_{lp2}(\omega)$ is

$$
f_{lp2}(\omega) = 2\omega c_k + (d_k^2 - c_k^2 - \omega^2) \times \left(\arctan\left(\frac{\omega - d_k}{c_k}\right) + \arctan\left(\frac{\omega + d_k}{c_k}\right)\right) + c_k d_k \left(\log\left(1 + \frac{(\omega - d_k)^2}{c_k^2}\right) - \log\left(1 + \frac{(\omega + d_k)^2}{c_k^2}\right)\right)
$$

Using Mathematica, we computed the definite integrals in (12) and verified the answers. Now that we have a closed-form solution for m_{lp}, we can substitute (13) into (10) to obtain a rather complicated but differentiable expression for the deviation from linear phase.

3.3. Filter Quality

The quality factor measures the relative distance of a filter pole from the imaginary frequency axis. The lower the quality factor, the less likely that the pole will cause oscillations in the output. The quality factor Q_k for the kth second-order section with conjugate poles $a_k \pm j b_k$ (with $a_k < 0$) and the effective overall quality factor Q_{eff} are

$$Q_k = \frac{\sqrt{a_k^2 + b_k^2}}{-2a_k} \qquad Q_{\text{eff}} = \left(\prod_{k=1}^{n} Q_i \right)^{\frac{1}{n}} \tag{14}$$

where $Q_k, Q_{\text{eff}} \geq 0.5$. $Q_k = 0.5$ corresponds to a double real-valued pole ($b_k = 0$), and $Q_k = \infty$ corresponds to an ideal oscillator ($a_k = 0$). We define Q_{eff} as the geometric mean of the quality factors, and other measures could be used. We use $Q_{\text{eff}} - 0.5$ to measure the filter quality.

3.4. Peak Overshoot in the Step Response

From the step response, we can numerically compute the peak overshoot and the time t_{peak} at which it occurs. In order to make the peak overshoot calculation differentiable, this section derives an analytic expression that approximates t_{peak} in terms of the pole-zero locations. The derivation assumes that there are no multiple poles.

The Laplace transform of the step response is

$$\begin{aligned} \frac{H(s)}{s} = \; & \frac{1}{s} \left[\prod_{k=1}^{n} \frac{a_k^2 + b_k^2}{s^2 - 2a_k s + a_k^2 + b_k^2} \right] \times \\ & \left[\prod_{k=1}^{n} \frac{s^2 - 2c_k s + c_k^2 + d_k^2}{c_k^2 + d_k^2} \right] \end{aligned} \tag{15}$$

Assuming no duplicate poles, partial fractions yields

$$\frac{H(s)}{s} = \left[\frac{A}{s} + \sum_{k=1}^{n} \frac{C_k s + D_k}{s^2 - 2a_k s + a_k^2 + b_k^2} \right] \tag{16}$$

$$\begin{aligned} C_k &= 2|B_k| \cos(\angle B_k) \\ D_k &= -2|B_k| \left(a_k \cos(\angle B_k) + b_k \sin(\angle B_k) \right) \\ B_k &= \left[H(s)(s - p_k) \right]_{s=p_k} = |B_k| e^{j \angle B_k} \\ A &= \left[H(s) \times s \right]_{s=0} = 1 \end{aligned}$$

$|B_k|$ and $\angle B_k$ can be expressed as real-valued differentiable functions of the pole and zero locations.

After inverse transforming (16), the step response is

$$h_{\text{step}}(t) = 1 + \sum_{k=1}^{n} \mathrm{e}^{a_k t} \left[C_k \cos(b_k t) + \left(\frac{D_k + C_k a_k}{b_k} \right) \sin(b_k t) \right] \tag{17}$$

By analyzing the kth term in the summation in (16), the kth peak overshoot occurs at time

$$t_{\text{peak}}^k = -\frac{1}{b_k} \left[\arctan \left(\frac{(D_k + 2C_k a_k) b_k}{C_k (a_k^2 - b_k^2) + D_k a_k} \right) + \pi \right] \tag{18}$$

We construct the following function to approximate t_{peak} for the purposes of computing derivatives:

$$t_{\text{peak}} \approx \frac{1}{n} \sum_{k=1}^{n} t_{\text{peak}}^k \;\Rightarrow\; t_{\text{peak}} = \beta \frac{1}{n} \sum_{k=1}^{n} t_{\text{peak}}^k \tag{19}$$

Here, β is set to the true value of t_{peak} (found numerically) divided by the approximation $\frac{1}{n} \sum_{k=1}^{n} t_{\text{peak}}^k$. We validated (19) using the SQP routine on several designs. We measure the peak overshoot cost by using $(h_{\text{step}}(t_{\text{peak}}) - 1)^2$.

4. CONSTRAINTS

This section discusses two sets of constraints. The first specifies the magnitude response, quality, and peak overshoot, and the second prevents numerical instabilities in the computations. We sample the magnitude response at a set of passband frequencies $\{\omega_i\}$ and stopband frequencies $\{\omega_l\}$:

$$1 - \delta_p \leq |H(j\omega_i)| \leq 1, \forall i \quad \text{ and } \quad |H(j\omega_l)| \leq \delta_s, \forall l \tag{20}$$

We compute the maximum overshoot by finding the maximum value of step response in (17) by searching over $t \in [\min_k t_{\text{peak}}^k, \max_k t_{\text{peak}}^k]$. Before finding the gradient of this constraint, we substitute the analytic approximation for t_{peak}, given by (19), into (17).

When the analog filter is implemented, the second-order sections will typically be cascaded in order of ascending quality factors. The earlier sections will attenuate input signals so as to minimize the oscillatory behavior of the final sections. The implementation technology imposes an upper limit on the quality factors, Q_{max}. For macro components, we set Q_{max} to 10 for $\omega_{p2} < 2\pi(10)$ kHz, and 25 otherwise:

$$\frac{\sqrt{a_k^2 + b_k^2}}{-2a_k} < Q_{max} \quad \text{for} \quad k = 1 \ldots n \tag{21}$$

Since a_k and c_k appear in the denominator in (2), (4), and (13), and b_k appears in the denominator in (17), we constrain these negative-valued parameters to be a neighborhood away from zero:

$$\begin{aligned} a_k < -\epsilon_{div} < 0 \quad & \text{for} \quad k = 1 \ldots n \\ b_k < -\epsilon_{div} < 0 \quad & \text{for} \quad k = 1 \ldots n \\ c_l < -\epsilon_{div} < 0 \quad & \text{for} \quad l = 1 \ldots r \end{aligned}$$

where ϵ_{div} is 2.2204×10^{-14} for MATLAB. To ensure the numerical stability of the denominators of $|B_k|$ and $\angle B_k$ in (16),

$$\sqrt{a_k - a_m} > \epsilon_{div} \quad \text{for } k = 1 \ldots n \text{ and } m = k+1 \ldots n$$

These constraints are analogous to preventing duplicate poles and poles spaced too closely to one another.

5. AN EXAMPLE FILTER DESIGN

We will minimize the peak overshoot and deviation from linear phase of a lowpass filter. The specifications on the magnitude response are $\omega_p = 20$ rad/sec with $\delta_p = 0.21$ and $\omega_s = 30$ rad/sec with $\delta_s = 0.31$. In the objective function, we weight the linear phase cost by 0.1 and overshoot cost by 1. The optimization took 13 seconds to run using MATLAB 5 on a 167 MHz Ultrasparc workstation. The non-negative objective function is reduced from an initial value of 2.87 to 4.33×10^{-5}. Table 1 and 2 list the initial and final poles and zeros, respectively. Figure 2 plots the frequency and step responses for the initial and final filters. Figure 2 illustrates that the optimization procedure effectively trades off transition bandwidth in the magnitude response for more linear phase in the passband and a lower overshoot. The peak overshoot is reduced from 25% to 10%.

Q	Poles	Zeros
1.7	$-5.3553 \pm j16.9547$	$\pm j20.2479$
61	$-.1636 \pm j19.9899$	$\pm j28.0184$

Table 1: Pole-zero locations for the initial filter

Q	Poles	Zeros
0.68	$-11.4343 \pm j10.5092$	$-3.4232 \pm j28.6856$
10	$-1.0926 \pm j21.8241$	$-1.2725 \pm j35.5476$

Table 2: Pole-zero locations for the optimized filter

6. CONCLUSION

We have developed a formal, extensible framework for optimizing multiple behavioral and implementation properties of analog filter designs. We have implemented the framework as a set of Mathematica programs that generate MATLAB programs to perform the optimization. Both the algebraic derivations and programming tasks would be nearly impossible for a human to carry out correctly. By performing both processes together, we can validate that the assumptions in the algebraic derivations are legitimate and that the source code is generated properly. Furthermore, the algebraic abstraction empowers the researcher to create new filter design programs by simply redefining the cost function— our software will take care of recomputing the derivatives and regenerating the source code.

7. REFERENCES

[1] T. Saramaki and K.-P. Estola, "Design of linear-phase partly digital anti-aliasing filters," in *Proc. IEEE Int. Conf. Acoust., Speech, and Signal Processing*, (Tampa, FL), Mar. 1985.

[2] M. Lutovac, D. V. Tosic, and B. L. Evans, "Algorithm for symbolic design of elliptic filters," in *Int. Workshop on Symbolic Methods and Applications to Circuit Design*, (Leuven, Belgium), pp. 248–251, Oct. 1996.

[3] S. Wright, "Convergence of SQP-like methods for constrained optimization," *SIAM Journal on Control and Optimization*, vol. 27, pp. 13–26, Jan. 1989.

[4] K. Schittkowski, "NLPQL: A Fortran subroutine solving constrained nonlinear programming problems," *Annals of Operations Research*, vol. 5, no. 1-4, pp. 485–500, 1986.

[5] S. Lawson and T. Wicks, "Improved design of digital filters satisfying a combined loss and delay specification," *IEE Proceedings G: Circuits, Devices and Systems*, vol. 140, pp. 223–229, June 1993.

[6] B. L. Evans, D. R. Firth, K. D. White, and E. A. Lee, "Automatic generation of programs that jointly optimize characteristics of analog filter designs," in *Proc. of European Conference on Circuit Theory and Design*, (Istanbul, Turkey), pp. 1047–1050, Aug. 1995.

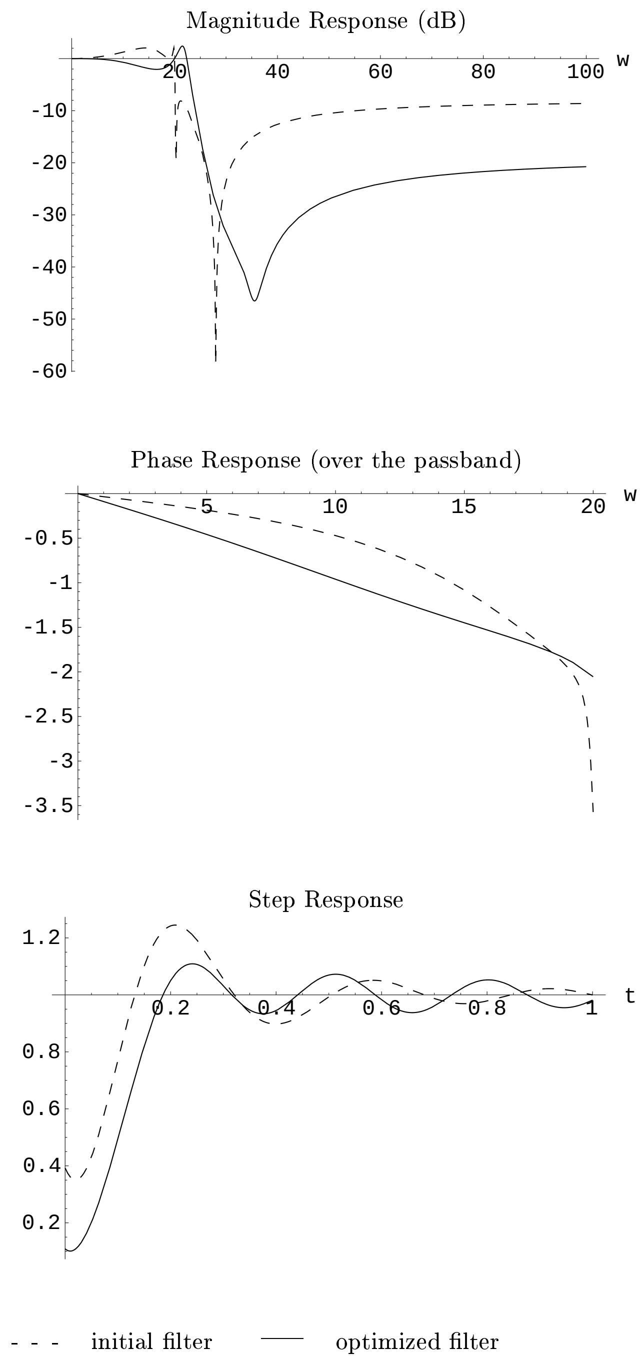

Figure 2: Fourth-order lowpass filter with user magnitude specifications $\omega_p = 20$ rad/s, $\delta_p = 0.21$, $\omega_s = 30$ rad/s, and $\delta_s = 0.31$. The initial filter is an elliptic filter design, and the final filter is optimized for phase and step response. We are trading linear phase response over the passband and peak overshoot in the step response for magnitude response, while keeping the magnitude response within specification. For the optimization, we set the maximum quality factor $Q_{\max}$ to be 10. Even though the initial guess is infeasible because its maximum Q value is 61, the SQP procedure in Matlab adjusted the initial guess to be a feasible solution.