Advanced Digital IIR Filter Design

Dejan V. Tošić[*] Miroslav D. Lutovac[†] Brian L. Evans[‡]

Abstract

Classical filter design techniques return only one design from an infinite collection of alternative designs, or fail to design filters when solutions exist. These classical techniques hide a wealth of alternative filter designs that are more robust when implemented in digital hardware and embedded software. In this paper, we present (1) case studies of optimal digital IIR filters that cannot be designed with classical techniques, and (2) the formal, mathematical framework that underlies their solutions. We have automated the advanced filter design techniques in software.

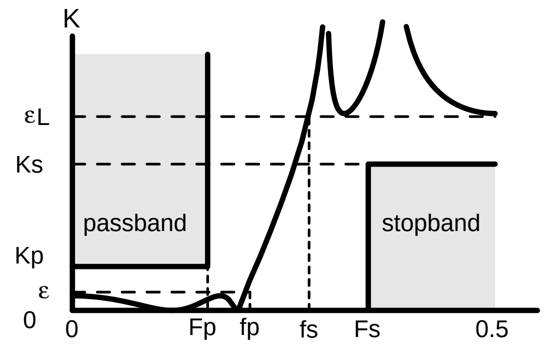

Figure 1: Characteristic function.

1 Introduction

In designing digital IIR filters, one generally relies on canned software routines or mechanical table-oriented procedures. The primary reason for these "black box" approaches is that the approximation theory that underlies filter design includes complex mathematics. Unfortunately, conventional approaches return only one design, thereby hiding a wealth of alternative filter designs that are more robust when implemented in digital hardware and embedded software. In addition, conventional approaches may fail to find a filter when in fact one exists.

We develop advanced design techniques to find a comprehensive set of optimal designs to represent the infinite solution space. The optimal designs include filters that have minimal order, minimal quality factors, minimal complexity, minimal sensitivity to pole-zero locations, minimal deviation from a specified group delay, approximate linear phase, and minimized peak overshoot. The design space also includes digital filters with power-of-two coefficients. We base our approach on formal, mathematical properties of Jacobi elliptic functions [1]. We automate these advanced filter design techniques in software [2, 3].

The key observations underlying advanced filter design are that (1) many designs satisfy the same user specification; (2) Butterworth and Chebyshev IIR Filters are special cases of Elliptic IIR Filters; and (3) minimum-order filters may not be as efficient to implement as some higher-order filters.

First, we introduce straightforward procedures to map the filter specification into a design space, i.e. a set of ranges for parameters that we use in the filter design. We search this design space for the optimum solution according to given criteria, such as minimal quantization error. Second, we present several new case studies of optimal digital IIR filters that cannot be designed with classical techniques, and the formal, mathematical framework that underlies their solutions.

2 Design space

We focus our attention on a lowpass filter that serves as the basis for the design of any lowpass, highpass, bandpass, or bandstop filter. First, we map the user specification into a characteristic-function specification S_K, Fig. 1, to provide a clearer relationship between the design parameters and the specification

$$S_K = \{F_p, F_s, K_p, K_s\} \tag{1}$$

Next, we identify the design parameters. Finally, we calculate the limits of the design parameters. The symbol F_p designates the passband edge frequency, and F_s designates the stopband edge frequency. We use the term "frequency" as a short form for "digital frequency." Frequency is a dimensionless quantity ranging from 0 to $\frac{1}{2}$.

An infinite number of characteristic functions that fit S_K exist. We consider the *elliptic function approximation*, because it fulfills the requirements

[*]School ECE, University of Belgrade, Yugoslavia
[†]IRITEL, R&D Institute, Belgrade, Yugoslavia
[‡]The Dept. ECE, The University of Texas, Austin, USA

with the minimal transfer function order. The minimal order can often lead to the most economical solution (the minimal number of multiplications).

The *prototype elliptic approximation*, K_e, is an nth-order rational function in the real variable x

$$K_e(x) = \epsilon |R(n, \xi, x)| \tag{2}$$

where R, referred to as the *rational elliptic function*, satisfies the conditions

$$\begin{array}{rcll} 0 & \leq & |R(n,\xi,x)| \leq 1, & |x| \leq 1 \\ L(n,\xi) & \leq & |R(n,\xi,x)|, & |x| \geq \xi \end{array} \tag{3}$$

and L is the minimal value of the magnitude of R for $|x| \geq \xi$

$$L(n,\xi) = |R(n,\xi,\xi)| \tag{4}$$

The normalized transition band $1 < x < \xi$ is defined by

$$1 < |R(n,\xi,x)| < L(n,\xi), \qquad 1 < |x| < \xi \tag{5}$$

The parameter ξ is called the *selectivity factor*.

The parameter ϵ determines the maximal variation of K_e in the normalized passband $0 \leq x \leq 1$

$$0 \leq K_e(x) \leq \epsilon, \qquad |x| \leq 1 \tag{6}$$

and is called the *ripple factor*.

The *elliptic approximation*, $K(f)$, is a rational function in frequency f, Fig. 1,

$$K(f) = K_e(x), \qquad x = \frac{\tan(\pi f)}{\tan(\pi f_p)} \tag{7}$$

where f_p is a design parameter that we call the *actual passband edge*. Traditionally, it has been set to $f_p = F_p$.

The four dimensionless quantities, n, ξ, ϵ, and f_p, are collectively referred to as *design parameters* and can be expressed as a list of the form

$$D = \{n, \xi, \epsilon, f_p\} \tag{8}$$

Each of the listed parameters can take a value from a continuous range (ξ, ϵ, f_p) or discrete range (n) of numbers. The order n is also referred to as the *filter order*. The elliptic function approximation provides the minimal order, $n_{min} = n_{ellip}$, for a given specification. The maximal order, from the practical viewpoint, can be assumed to be $n_{max} = 2n_{min}$.

The selectivity factor, ξ, falls within the limits which are found by solving the equations [3]

$$R(n,\xi,\xi) = \frac{K_s}{K_p} \Rightarrow \xi_{min} = \xi_{min}(n) \tag{9}$$

$$\left.\begin{array}{l} R\left(n, \xi, \dfrac{\tan(\pi F_s)}{\tan(\pi F_p)}\right) = \dfrac{K_s}{K_p} \\ \xi > \dfrac{\tan(\pi F_s)}{\tan(\pi F_p)} \end{array}\right. \Rightarrow \xi_{max} = \xi_{max}(n) \tag{10}$$

The ripple factor ϵ can take a value from the range

$$\begin{array}{c} \epsilon_{min} = \dfrac{K_s}{L(n,\xi)} = \epsilon_{min}(n,\xi) \\ \epsilon_{max} = K_p \\ \epsilon_{min} \leq \epsilon \leq \epsilon_{max} \end{array} \tag{11}$$

The actual passband edge, f_p, can take a value from the interval

$$\begin{array}{rcl} f_{p,min} & = & \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{\tan(\pi F_s)}{\xi_{max}}\right) \\ f_{p,max} & = & \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{\tan(\pi F_s)}{\xi_{min}}\right) \\ & & f_{p,min} \leq f_p \leq f_{p,max} \end{array} \tag{12}$$

The set of all quadruples $D = \{n, \xi, \epsilon, f_p\}$, satisfying the constraints $\{n_{min} \leq n \leq n_{max}, \xi_{min} \leq \xi \leq \xi_{max}, \epsilon_{min} \leq \epsilon \leq \epsilon_{max}, f_{p,min} \leq f_p \leq f_{p,max}\}$, is called the *design space*.

$$D_S = \{D_{S,n}\}|_{n = n_{min}, n_{min}+1, \ldots, n_{max}} \tag{13}$$

$$D_{S,n} = \left\{ \begin{array}{ccccc} & & n & & \\ \xi_{min}(n) & \leq & \xi & \leq & \xi_{max}(n) \\ \epsilon_{min}(n,\xi) & \leq & \epsilon & \leq & K_p \\ f_{p,min}(n) & \leq & f_p & \leq & f_{p,max}(n) \end{array} \right.$$

Since the integer order n takes only discrete numeric values, it is more convenient to express the design space, D_S, as a list of subspaces, $D_{S,n}$, where

$$\begin{array}{ccccc} 0 & < & \epsilon_{min}(n+1) & < & \epsilon_{min}(n) \\ 1 & < & \xi_{min}(n+1) & < & \xi_{min}(n) \\ \xi_{max}(n) & < & \xi_{max}(n+1) & \leq & +\infty \\ 0 & \leq & f_{p,min}(n+1) & < & f_{p,min}(n) \\ f_{p,max}(n) & < & f_{p,min}(n+1) & \leq & \frac{1}{2} \end{array}$$

3 Basic design alternatives

This section presents our case studies of a comprehensive set of design alternatives based on the design space. It is understood that the rational elliptic function can be readily constructed for a given set of design parameters [1]. Usually, the designer selects the minimal order $n = n_{min}$. The design alternatives that follow are general and valid for any n from the design space. We define six basic designs (denoted by D1, D2, D3a, D3b, D4a, D4b) by choosing the design parameters as follows:

$$
\begin{array}{llll}
 & \xi & \epsilon & f_p \\
\mathrm{D1} & \dfrac{\tan(\pi F_s)}{\tan(\pi F_p)} & K_p & F_p \\
\mathrm{D2} & \dfrac{\tan(\pi F_s)}{\tan(\pi F_p)} & \dfrac{K_s}{L} & F_p \\
\mathrm{D3a} & \xi_{min} & K_p & F_p \\
\mathrm{D3b} & \xi_{min} & K_p & \dfrac{\tan^{-1}\left(\frac{\tan(\pi F_s)}{\xi_{min}}\right)}{\pi} \\
\mathrm{D4a} & \xi_{\max} & K_p & F_p \\
\mathrm{D4b} & \xi_{\max} & \dfrac{K_s}{L} & \dfrac{\tan^{-1}\left(\frac{\tan(\pi F_s)}{\xi_{max}}\right)}{\pi}
\end{array}
\tag{14}
$$

Characteristic function for each design is shown in Fig. 2. Design D1 has higher attenuation in the stopband than it is required by the specification. We choose this design when we prefer to achieve as large attenuation as possible in the stopband.

Design D2 has lower attenuation in the passband than it is required by the specification. We choose this design when we prefer to achieve as low attenuation as possible in the passband. Also, this design is suitable when filter finite wordlength effects can significantly change the magnitude in the passband.

Design D3a has the sharpest magnitude response. When undesired signals exist in the transition region we may prefer design D3a, because it rejects the undesired signals as much as possible. Design D3b has the sharpest magnitude response (the same as D3a). When the desired signals exist in the transition region we may prefer the design D3b, because it attenuates the desired signals as low as possible.

Design D4a (like the design D1) has higher attenuation in the stopband than it is required by the specification, except at the stopband edge frequency. We choose this design when we prefer to achieve as large attenuation as possible in the stopband. Design D4b (like the design D2) has lower attenuation in the passband than it is required by the specification, except at the passband edge frequency. We choose this design when we prefer to achieve as low attenuation as possible in the passband. A disadvantage of D3a, D3b, D4a and D4b is lack of any attenuation margin. Any imperfection, usually in implementation step (like coefficient quantization), can violate the specification.

4 Visualization of design space

A lowpass filter will be designed to meet the attenuation specification
$S_A = \{F_p, F_s, A_p, A_s\} = \{0.2, 0.212, 0.2\,\mathrm{dB}, 40\,\mathrm{dB}\}$
where A_p designates the maximum attenuation in passband, and A_s is the minimum attenuation in stopband.

We will consider the mapped specification
$S_K = \{F_p, F_s, K_p, K_s\} = \{0.2, 0.212, 0.2171, 100\}$

We have calculated the minimal filter order $n_{\min} = 8$. Next, the range of ξ, ϵ, f_p, and f_s has been determined for $n \geq 8$, as shown in Table 1. The design subspace is shown in Fig. 3.

Table 1: Design space $S_A = \{0.2, 0.212, 0.2, 40\}$.

n	8	9	10	11	13
ϵ_{min}	0.063	0.0143	10^{-4}	10^{-6}	10^{-11}
ϵ_{max}	0.217	0.217	0.217	0.217	0.217
$f_{p,min}$	0.198	0.196	0.193	0.188	0.171
$f_{p,max}$	0.206	0.209	0.210	0.211	0.212
$f_{s,min}$	0.206	0.203	0.202	0.201	0.200
$f_{s,max}$	0.214	0.216	0.220	0.225	0.243

Classical approximations will be unacceptable: the order of the Chebyshev and Inverse Chebyshev type filter is high ($n_{cheb} = 18$), while the order of the Butterworth filter is extremely high ($n_{butt} = 79$).

The minimal filter order, $n = n_{min}$, implies a small range for design parameters and the optimization of the filter behavior can be ineffective. It is also worth noticing that increasing the filter order, $n > n_{min}$, does not necessarily lead to a better solution. However, in many practical filter designs the improvement was considerable [3, 4].

The design parameters, the actual stopband edge, the maximal attenuation in the passband, and the minimal attenuation in the stopband, are summarized in Table 2.

Table 2: Digital filter design summary, $n = 8$.

D	ξ	ϵ	f_p	f_s	a_p(dB)	a_s
1	1.08155	0.217	0.2	0.212	0.2	49
2	1.08155	0.079	0.2	0.212	0.03	40
3a	1.04285	0.217	0.2	0.206	0.2	40
3b	1.04285	0.217	0.205	0.212	0.2	40
4a	1.09245	0.217	0.2	0.214	0.2	51
4b	1.09245	0.063	0.198	0.212	0.02	40

By increasing the filter order the design D4a arrives at the Chebyshev type approximation, for $f_s = \frac{1}{2}$. Alternatively, for the same order, and $f_p = 0$, the design D4b yields an Inverse Chebyshev type filter. When the filter order is equal to the order of the Butterworth type filter, with $f_p = 0$ and $f_s = \frac{1}{2}$, the elliptic approximation transforms into the Butterworth approximation. This means

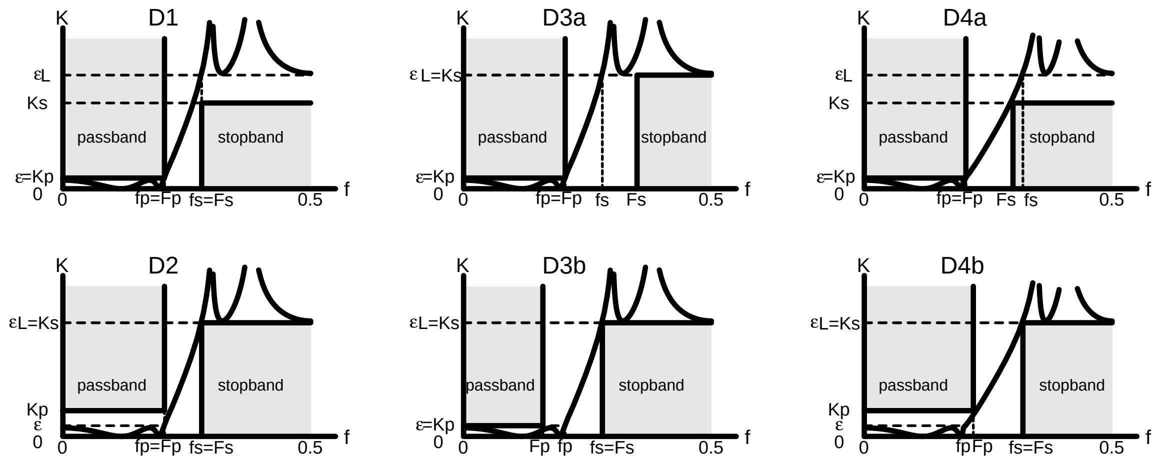

Figure 2: Designs D1, D2, D3a, D3b, D4a, D4b.

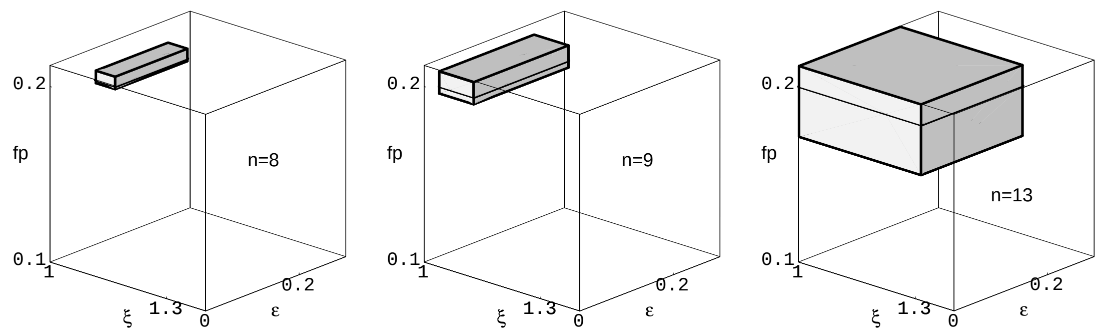

Figure 3: Design subspace for $n = 8$, $n = 9$ and $n = 13$.

that the classical filter types, Chebyshev, Inverse Chebyshev and Butterworth, are just special cases of the elliptic filters, and are contained within the design space D_S.

5 Conclusion

We present several case studies of optimal digital IIR filter design, and show that conventional approaches to filter design either return only one design thereby hiding a wealth of robust alternatives or fail to find a design when a design exists. We develop advanced design techniques to find a comprehensive set of optimal designs to represent the infinite solution space. The optimal designs include filters that have minimal order, minimal complexity, minimal number of multipliers, power-of-two multipliers, etc. We have observed that many designs satisfy the same user specification, and that minimum-order filters may not be as efficient to implement as some higher-order filters. This approach we have programmed in *Mathematica*.

References

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