EMQF filter design in MATLAB

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Abstract— **In this paper we present a straightforward procedure of EMQF (Elliptic Minimal Q-Factor) analog filter design and detailed step-by-step design algorithm. The main feature of EMQF filter is that the magnitude response in the passband and stopband is insensitive to variations of the element values and, thus, the implementations in analog circuits are very robust with higher element tolerances.**

I. Introduction

There exists a special class of elliptic filters called EMQF filters (Elliptic Minimal Q-Factor), which is of prime importance in many practical applications. The principle feature of the minimal Q-factor filters is that their transfer function poles lay on a circle in the complex s-plain [1]. A very selective specifications could be fulfilled with the EMQF filter order that is slightly higher than the minimal order of the elliptic filter [2], [3], [4], [5]. In addition, the EMQF filter design features low Q-factors and reduced overall sensitivity. The maximal magnitude-response deviation, due to element tolerances, is in the transition region. Practically, the magnitude response in the passband and stopband is insensitive to variations of the value of filter elements [6], [7], [8], [9], [10], [11].

Classical analog filter design techniques return only one design from an infinite collection of alternative designs, or fail to design filters when solutions exist. These classical techniques hide a wealth of alternative filter designs that are more robust when implemented in analog circuits. In this paper, we present (1) an algorithm of EMQF analog filter that cannot be designed with classical techniques, and (2) the formal, mathematical framework that underlies the algorithmic steps. We have automated the EMQF filter design technique in software, so we present detailed step-by-step design algorithm.

II. Notation

Q-factor of a transfer function pole s_i is defined as

$$Q_i = -\frac{|s_i|}{2 \text{ Re } s_i} \tag{1}$$

The minimal value of the Q-factor is 1/2 and it occurs when the pole is real. The Q-factor takes the infinite value for poles on the imaginary axis.

We review the list of symbols that we use in formulas and procedures when designing EMQF filters:

$a(f)$ — attenuation (dB)
a_p — maximum passband attenuation of designed filter
a_s — minimum stopband attenuation of designed filter
A_p — maximum passband attenuation in spec (dB)
A_s — minimum stopband attenuation in spec (dB)
$\mathrm{cd}(u,k)$ — Jacobi elliptic cd function
f — frequency (Hz)
f_{nQ} — normalized frequency in EMQF filter design
f_p — passband edge frequency of designed filter (Hz)
F_p — passband edge frequency in spec (Hz)
f_s — stopband edge frequency of designed filter (Hz)
F_s — stopband edge frequency in spec (Hz)
$\mathcal{H}_{\mathrm{minQ}}(n,\xi,p)$ — EMQF normalized lowpass transfer function
$H(s)$ — transfer function
i — index ($i = 1, 2, \ldots, n$) or iterator
j — the imaginary unit ($j = \sqrt{-1}$)
k — modulus of elliptic functions
$K_e(n,\xi,\epsilon,x)$ — elliptic characteristic function
$K_J(k)$ — complete elliptic integral of first kind
K_p — characteristic function passband spec
K_s — characteristic function stopband spec
$L(n,\xi)$ — discrimination factor
n — transfer function order (order for short)
$n_{ellip}(F_p, F_s, K_p, K_s)$ — minimum elliptic order
$n_{minQ}(F_p, F_s, K_p, K_s)$ — minimum order of EMQF filter
p — normalized complex frequency
$Q_{\mathrm{minQ}}(n,\xi,i)$ — quality factor of ith pole of EMQF filter
$R(n,\xi,x)$ — elliptic rational function
s — complex frequency (rad/s)
S_A — attenuation-limit specification (spec for short)
x, y — dimensionless variables
$X(n,\xi,i)$ — ith zero of elliptic rational function
ϵ — ripple factor
ξ — selectivity factor
$\lfloor x \rfloor$ — integer such that $x \le \lfloor x \rfloor < x+1$
$\underset{x_1<x<x_2}{\mathrm{FindRoot}}\ \{\, F(x) = G(x) \,\}$ — find real x over interval $x_1 < x < x_2$ by solving $F(x) = G(x)$
$\underset{\substack{x_1<x<x_2\\ y_1<y<y_2}}{\mathrm{FindRoot}} \left\{ \begin{array}{l} F_1(x) = G_1(x) \\ F_2(x) = G_2(x) \end{array} \right\}$ — find real x over interval $x_1 < x < x_2$ and real y over interval $y_1 < y < y_2$ by solving set of equations $\{F_1(x) = G_1(x),\ F_2(x) = G_2(x)\}$.

III. EMQF Design Equations and Procedures

In this section we summarize all EMQF filter design equations, formulas and procedures that are based on Jacobi elliptic functions.

The special mathematical function, $K_J(k)$, is used in the design of elliptic filters

$$K_J(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{2}$$

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The basic functions for the EMQF filter design are $X(n,\xi,i)$ and $Q_{\mathrm{minQ}}(n,\xi,i)$, as follows:

$$\boxed{\begin{array}{l} X(n,\xi,i) = -\mathrm{cd}\left(\dfrac{2i-1}{n}K_J(\dfrac{1}{\xi}),\dfrac{1}{\xi}\right) \\ X(n,\xi,(n+1)/2) = 0, \quad n \text{ odd} \\ X(n,\xi,1) < X(n,\xi,2) < \ldots < X(n,\xi,n) \end{array}} \tag{3}$$

$$\boxed{Q_{\mathrm{minQ}} \quad = \frac{\xi + X(n,\xi,i)^2}{2\sqrt{1-X(n,\xi,i)^2}\sqrt{\xi^2 - X(n,\xi,i)^2}}} \tag{4}$$

The essential functions for the elliptic approximation are $L(n,\xi)$, $R(n,\xi,x)$, and $K_e(n,\xi,\epsilon,x)$:

$$\boxed{\begin{array}{l} n \text{ even}, \quad n = 2,4,\ldots \\ r_0 = \dfrac{\prod_{i=1}^{n/2}\left(1-X^2(n,\xi,i)\right)}{\prod_{i=1}^{n/2}\left(1-\dfrac{\xi^2}{X^2(n,\xi,i)}\right)} \\ R(n,\xi,x) \quad = \dfrac{1}{r_0}\dfrac{\prod_{i=1}^{n/2}\left(x^2-X^2(n,\xi,i)\right)}{\prod_{i=1}^{n/2}\left(x^2-\dfrac{\xi^2}{X^2(n,\xi,i)}\right)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ n \text{ odd}, \quad n = 1,3,\ldots \\ r_0 = \dfrac{\prod_{i=1}^{(n-1)/2}\left(1-X^2(n,\xi,i)\right)}{\prod_{i=1}^{(n-1)/2}\left(1-\dfrac{\xi^2}{X^2(n,\xi,i)}\right)} \\ R(n,\xi,x) \quad = \dfrac{x}{r_0}\dfrac{\prod_{i=1}^{(n-1)/2}\left(x^2-X^2(n,\xi,i)\right)}{\prod_{i=1}^{(n-1)/2}\left(x^2-\dfrac{\xi^2}{X^2(n,\xi,i)}\right)} \end{array}} \tag{5}$$

$$\boxed{K_e\left(n,\xi,\epsilon,x\right) = \epsilon\left|R\left(n,\xi,x\right)\right|} \tag{6}$$

$$\boxed{\begin{array}{l} n \text{ even}, \quad n = 2,4,\ldots \\ L(n,\xi) = \dfrac{1}{\xi^n}\dfrac{\prod_{i=1}^{n/2}\left(\xi^2-X^2(n,\xi,i)\right)^2}{\prod_{i=1}^{n/2}\left(1-X^2(n,\xi,i)\right)^2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ n \text{ odd}, \quad n = 1,3,\ldots \\ L(n,\xi) = \dfrac{1}{\xi^{n-2}}\dfrac{\prod_{i=1}^{(n-1)/2}\left(\xi^2-X^2(n,\xi,i)\right)^2}{\prod_{i=1}^{(n-1)/2}\left(1-X^2(n,\xi,i)\right)^2} \end{array}} \tag{7}$$

The EMQF normalized lowpass transfer function $\mathcal{H}_{\mathrm{minQ}}(n,\xi,\epsilon,p)$

$$\boxed{\begin{array}{l} n \text{ even}, \quad n = 2,4,\ldots \\ g = \dfrac{\prod_{i=1}^{n/2}X(n,\xi,i)^2}{\sqrt{\xi^n}\sqrt{1+\dfrac{1}{L(n,a)}}} \\ \mathcal{H}_{\mathrm{minQ}} \quad = g\dfrac{\prod_{i=1}^{n/2}p^2+\dfrac{\xi^2}{X^2(n,\xi,i)}}{\prod_{i=1}^{n/2}p^2+p\dfrac{\sqrt{\xi}}{Q_{\mathrm{minQ}}(n,\xi,i)}+\xi} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ n \text{ odd}, \quad n = 1,3,\ldots \\ g = \dfrac{1}{p+\sqrt{\xi}}\dfrac{\prod_{i=1}^{\frac{n-1}{2}}X(n,\xi,i)^2}{\sqrt{\xi^{n-2}}\sqrt{1+\dfrac{1}{L(n,a)}}} \\ \mathcal{H}_{\mathrm{minQ}} \quad = g\dfrac{\prod_{i=1}^{\frac{n-1}{2}}p^2+\dfrac{\xi^2}{X^2(n,\xi,i)}}{\prod_{i=1}^{\frac{n-1}{2}}p^2+p\dfrac{\sqrt{\xi}}{Q_{\mathrm{minQ}}(n,\xi,i)}+\xi} \end{array}} \tag{8}$$

Attenuation-limit lowpass specification

$$\boxed{S_A = \{F_p, F_s, A_p, A_s\}} \tag{9}$$

Characteristic-function-limit lowpass specification

$$\boxed{S_K = \{F_p, F_s, K_p, K_s\}} \tag{10}$$

$$\boxed{\begin{array}{l} K_p(A_p) = \dfrac{\sqrt{1-10^{-A_p/10}}}{10^{-A_p/20}} \\ K_s(A_s) = \dfrac{\sqrt{1-10^{-A_s/10}}}{10^{-A_s/20}} \end{array}} \tag{11}$$

We compute the minimal orders $n_{\mathrm{minQ}}\left(F_p,F_s,K_p,K_s\right)$ and $n_{ellip}\left(F_p,F_s,K_p,K_s\right)$

$$\boxed{\begin{array}{l} k = \dfrac{F_p}{F_s} \\ L \ = \dfrac{K_s}{K_p} \\ N = \dfrac{K_J\left(\sqrt{1-\dfrac{1}{L^2}}\right)}{K_J\left(\dfrac{1}{L}\right)} \\ D = \dfrac{K_J\left(\sqrt{1-k^2}\right)}{K_J\left(k\right)} \\ n_{ellip} \quad = \left\lceil \dfrac{N}{D} \right\rceil \end{array}} \tag{12}$$

$$
\boxed{\begin{array}{l}
i = n_{ellip}\left(F_p, F_s, K_p, K_s\right) \\
\xi_1 = \dfrac{F_p}{F_s} \\
\xi_2 = \xi_1^2 \\
\xi_i = x \left| \underset{\xi_1 < x < \xi_2}{\text{FindRoot}} \sqrt{L(i,x)} = K_s \right. \\
\boxed{\begin{array}{l}
\text{While } K_e(i, \xi_i, \dfrac{1}{K_s}, \dfrac{F_s}{F_p}\xi_i) > K_p \\
i = i+1 \\
\xi_2 = \xi_i \\
\xi_i = x \left| \underset{\xi_1 < x < \xi_2}{\text{FindRoot}} \sqrt{L(i,x)} = K_s \right.
\end{array}} \\
n_{\text{minQ}} = i
\end{array}} \tag{13}
$$

The maximal selectivity factor $\xi_{\max}\left(n, F_p, F_s, K_p, K_s\right)$

$$
\boxed{\xi_{\max} = x \left| \underset{F_s/F_p < x < 10F_s/F_p}{\text{FindRoot}} R(n, x, \frac{F_s}{F_p}) = \frac{K_s}{K_p} \right.} \tag{14}
$$

The EMQF maximal selectivity factor $\xi_{\text{minQ}}\left(n, F_p, F_s, K_p, K_s\right)$, and the EMQF normalized frequency $f_{nQ}\left(n, F_p, F_s, K_p, K_s\right)$

$$
\xi_h = \xi_{\max}\left(n, F_p, F_s, K_p, K_s\right)
$$

$$
\xi_l = x \left| \underset{\sqrt{\xi_h \frac{F_s}{F_p}} < x < \xi_h}{\text{FindRoot}} K_e(n, x, \frac{1}{\sqrt{L(n,x)}}, x\frac{F_s}{F_p}) = K_p \right.
$$

$$
f_l = x \left| \underset{\sqrt[4]{\xi_l} < x < \sqrt{\xi_l}}{\text{FindRoot}} K_e(n, \xi_l, \frac{1}{\sqrt{L(n, \xi_l)}}, x) = K_p \right.
$$

$$
f_h = x \left| \underset{\sqrt[4]{\xi_h} < x < \sqrt{\xi_h}}{\text{FindRoot}} K_e(n, \xi_h, \frac{1}{\sqrt{L(n, \xi_h)}}, x) = K_p \right.
$$

$$
\begin{array}{l} \xi_{\text{minQ}} = x \\ f_{nQ} = y \end{array} \underset{\substack{\xi_l < x < \xi_h \\ f_l < y < f_h}}{\text{FindRoot}} \left\{ \begin{array}{l} K_e(n, x, \frac{1}{\sqrt{L(n,x)}}, y) = K_p \\ K_e(n, x, \frac{1}{\sqrt{L(n,x)}}, y\frac{F_s}{F_p}) = K_s \end{array} \right\} \tag{15}
$$

We use the above relations in the numerical design.

IV. EMQF Filter Design Algorithm

A detailed step-by-step procedure for computing the EMQF lowpass transfer function with the maximum selectivity follows:

1. Start from a specification and convert it into the characteristic-function-limit specification

$$
S_A = \{F_p, F_s, A_p, A_s\} \longmapsto S_K = \{F_p, F_s, K_p, K_s\} \tag{16}
$$

2. Compute the minimal order $n_{\text{minQ}}\left(F_p, F_s, K_p, K_s\right)$
3. Choose the EMQF order

$$
n \geq n_{\text{minQ}}\left(F_p, F_s, K_p, K_s\right) \tag{17}
$$

4. Compute the EMQF maximal selectivity factor

$$
\xi = \xi_{\text{minQ}}\left(n, F_p, F_s, K_p, K_s\right) \tag{18}
$$

and the corresponding EMQF normalized frequency

$$
f_{nQ}\left(n, F_p, F_s, K_p, K_s\right) \tag{19}
$$

5. Compute the EMQF ripple factor

$$
\epsilon = \frac{1}{\sqrt{L(n, \xi)}} \tag{20}
$$

6. Compute the EMQF actual passband edge

$$
f_p = \frac{F_p}{f_{nQ}\left(n, F_p, F_s, K_p, K_s\right)} \tag{21}
$$

7. Construct the normalized lowpass transfer function

$$
\mathcal{H}(n, \xi, \epsilon, p) \tag{22}
$$

8. Construct the EMQF lowpass transfer function

$$
H(s) = \mathcal{H}(n, \xi, \epsilon, \frac{s}{2\pi f_p}) \tag{23}
$$

EXAMPLE

For a given spec $S_A = \{3\,\text{Hz}, 4\,\text{Hz}, 0.2\,\text{dB}, 20\,\text{dB}\}$, we find the order $n_{\text{minQ}} = 4$, the selectivity factor $\xi_{\text{minQ}} = 1.412$, the ripple factor $\epsilon = 0.08686$, the actual passband edge $f_p = 2.852\,\text{Hz}$, the EMQF lowpass transfer function

$$
H(s) = \frac{b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}
$$

$$
\begin{array}{ll}
b_4 = 8.39043190470 \cdot 10^{-7} & a_4 = 9.69635689477 \cdot 10^{-6} \\
b_3 = 0 & a_3 = 3.71192253704 \cdot 10^{-4} \\
b_2 = 3.37168171565 \cdot 10^{-3} & a_2 = 0.0107556987695 \\
b_1 = 0 & a_1 = 0.168290719763 \\
b_0 = 1.98563643345 & a_0 = 1.99311235679
\end{array}
$$

The value of the highest pole Q-factor is $Q_{\text{minQ}} = 3.359$. The EMQF attenuation characteristic is shown in Fig. 1.

It should be noticed that the edge frequencies of the designed filter are not equal to the edge frequencies in spec, and also the passband attenuation is very small except at the passband edge frequency

$$
\begin{array}{lcl}
f_p = 2.852\,\text{Hz} & < & F_p = 3\,\text{Hz} \\
f_s = 4.026\,\text{Hz} & > & F_s = 4\,\text{Hz} \\
a_p = 0.03\,\text{dB} & \ll & A_p = 0.2\,\text{dB} \\
a_s = 21.2\,\text{dB} & > & A_s = 20\,\text{dB} \\
a(F_p) & = & A_p = 0.2\,\text{dB} \\
a(F_s) & = & A_s = 20\,\text{dB}
\end{array}
$$

V. Conclusion

In this paper we present a detailed step-by-step design algorithm of EMQF (Elliptic Minimal Q-Factor) analog filter. We have automated this algorithm in software (MATLAB). The software can be downloaded from `http://galeb.etf.bg.ac.yu/~tosic/afdhome.htm`.

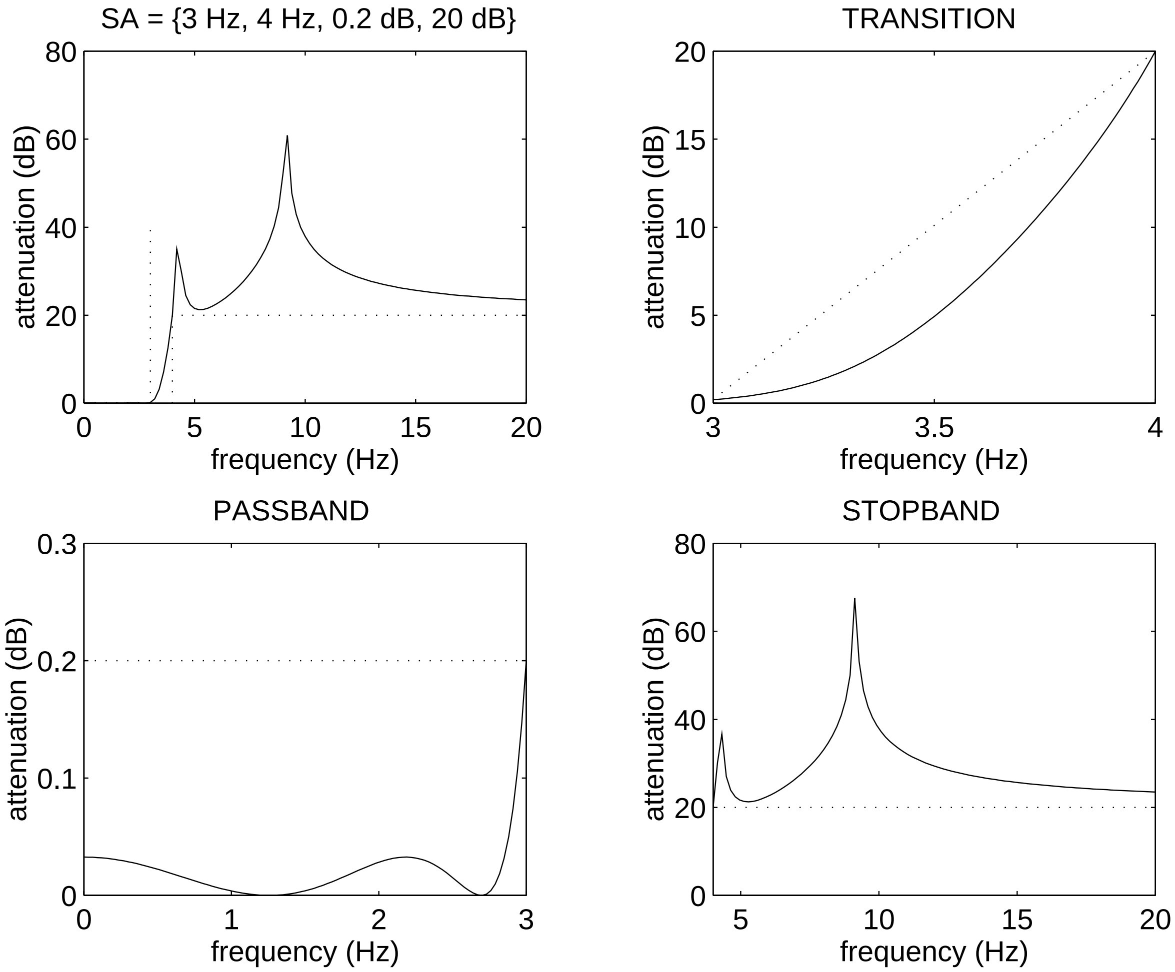

Fig. 1. Attenuation characteristic of an EMQF filter for $S_A = \{3\,\mathrm{Hz},\ 4\,\mathrm{Hz},\ 0.2\,\mathrm{dB},\ 20\,\mathrm{dB}\}$.

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