JOINT OPTIMIZATION OF MULTIPLE BEHAVIORAL AND IMPLEMENTATION PROPERTIES OF DIGITAL IIR FILTER DESIGNS

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ABSTRACT

This paper presents an extensible framework for the simultaneous constrained optimization of multiple properties of digital IIR filters. The framework optimizes the pole-zero locations for behavioral properties of magnitude and phase response, and the implementation property of quality factors, subject to constraints on the same properties. We formulate the constrained nonlinear optimization problem as a sequential quadratic programming (SQP) problem. SQP solvers are robust when provided formulas for the gradients of the objective function and constraints. We program Mathematica to compute the gradient formulas and convert the formulas into Matlab programs to perform the optimization. The automated approach eliminates errors in manipulating the algebraic equations and transcribing equations into software. The key contributions are (1) an automated, extensible, multicriteria digital IIR filter optimization framework, and (2) a novel filter design. We have released the source code on the Internet.

1. INTRODUCTION

Classical digital infinite impulse response (IIR) filters introduce significant phase distortion in output signals. This phase distortion is generally acceptable in single-speaker voice processing applications, such as speech coding and plain old telephone service, but generally unacceptable in digital communications, digital audio, and digital image processing. A conventional method to linearize the phase is to cascade allpass filters. This structure is clearly inefficient when compared to a unified design. Digital IIR filters may also suffer from numerical instability when implemented. Quality factors provide a technology-independent measure of the potential numerical instability.

Elliptic, Chebyshev and Butterworth digital IIR filter designs yield desirable behavioral properties subject to constraints on the magnitude response. Bessel filters exhibit low overshoot and nearly linear phase response over the passband, but poor magnitude response. Classical elliptic filter designs have minimal order. For a given implementation technology, minimal order filters may not be realizable or have minimal complexity [1]. Two modern methods for multiple criteria filter designs only handle a particular class of filters and do not consider quality factors [2, 3]. In [2], filters are designed as a parallel arrangement of allpass sections. Instead, we provide a framework to optimize and improve the filters generated by conventional methods.

This paper develops a framework for the simultaneous optimization of multiple user-specified criteria for lowpass digital IIR filter prototypes. We use the framework to design a digital IIR filter with nearly linear phase response over the passband and minimized quality factors, subject to constraints on the magnitude response. The framework requires an initial filter design. If one is not supplied by the user, then a conventional elliptic filter design will be generated. New properties may be added to the framework. The framework is an extension of our multicriteria analog IIR filter optimization framework [4].

We model the constrained nonlinear optimization problem as a sequential quadratic programming (SQP) problem. SQP requires that the objective function and constraints be real-valued and twice differentiable with respect to the free parameters [5]. The free parameters are the pole-zero locations. In order to avoid divergence that may occur when an SQP solver approximates the gradients of the objective function and constraints with respect to the free parameters [4], we supply closed-form gradients. We develop *Mathematica* software [6] to compute the gradients and translate the SQP formulation into MATLAB programs [7, 8] to perform the optimization. Constraints and objective functions may be added or modified in the framework. The *Mathematica* software would regenerate the MATLAB code.

2. NOTATION

We focus on lowpass even-order digital IIR filter specified by its n complex conjugate pole pairs, and m complex conjugate zero pairs. We denote the kth pole pair as $p_k = a_k e^{\pm j b_k}$, where $a_k < 1$ for stability, and the lth zero pair as $q_l = c_l e^{\pm j d_l}$. We assume that the filter is lowpass. The transfer function in the z-transform domain is

$$H(z) = A \frac{\prod_{l=1}^{m} \left(1 - 2z^{-1}\cos(d_l) + z^{-2}c_l^2\right)}{\prod_{k=1}^{n} \left(1 - 2z^{-1}\cos(b_l) + z^{-2}a_l^2\right)} \qquad (1)$$

This work was supported by US Defense Advanced Research Projects Agency (Grant DAAB07-97-C-J007) and a US National Science Foundation CAREER Award (Grant MIP-9702707).

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$$\left|H(e^{j\omega})\right| = A\frac{\prod_{l=1}^{m}\sqrt{(1+c_l{}^2-2c_l\cos(d_l-\omega))(1+c_l{}^2-2c_l\cos(d_l+\omega))}}{\prod_{k=1}^{n}\sqrt{(1+a_k{}^2-2a_k\cos(b_k-\omega))(1+a_k{}^2-2a_k\cos(b_k+\omega))}}$$

Figure 1: Magnitude response of a digital IIR filter as real-valued differentiable function, where ω denotes the digital frequency. The filter has m pairs of zeros and n pairs of poles. The lth zero pair is $c_l e^{\pm j d_l}$. The kth pole pair is $a_k e^{\pm j b_k}$.

where A is a constant that normalizes the magnitude response at DC to unity.

The magnitude response $|H(e^{j\omega})|$ as a real-valued function is shown in figure 1. The magnitude response is not differentiable at frequencies ω, where $|H(e^{j\omega})| = 0$, due to the square root operator. However, $|H(e^{j\omega})|^2$ is differentiable at all ω.

The unwrapped phase response is difficult to express as a function that is differentiable with respect to the pole-zero locations. We write the phase response as the sum of phase components from the poles and zeros. The phase component due to a single pole pair, p_k, within the unit circle can be expressed as a differentiable function:

$$\begin{aligned}\angle H_{p_k}(e^{j\omega}) = \quad & -\tan^{-1}\left(\frac{a_k\sin(\omega-b_k)}{1-a_k\cos(\omega-b_k)}\right) \\ & -\tan^{-1}\left(\frac{a_k\sin(\omega+b_k)}{1-a_k\cos(\omega+b_k)}\right)\end{aligned} \tag{2}$$

The phase component due to a single zero pair, q_l, is

$$\begin{aligned}\angle H_{q_l}(e^{j\omega}) = \quad & \tan^{-1}\left(\frac{c_l\sin(\omega-d_l)}{1-c_l\cos(\omega-d_l)}\right) \\ & +\tan^{-1}\left(\frac{c_l\sin(\omega+d_l)}{1-c_l\cos(\omega+d_l)}\right)\end{aligned} \tag{3}$$

If all zeros lie inside the unit circle, then (3) is continuous and differentiable with respect to zero locations and ω. For zeros outside or on the unit circle, i.e. $c_l \geq 1$, the phase is wrapped at $\omega = d_l$. The total phase response is

$$\angle H(e^{j\omega}) = \sum_{k=1}^{n}\angle H_{p_k}(e^{j\omega}) + \sum_{l=1}^{m}\angle H_{q_l}(e^{j\omega}) \tag{4}$$

In this paper, Q represents quality factor, σ represents deviation, and m represents slope of a line.

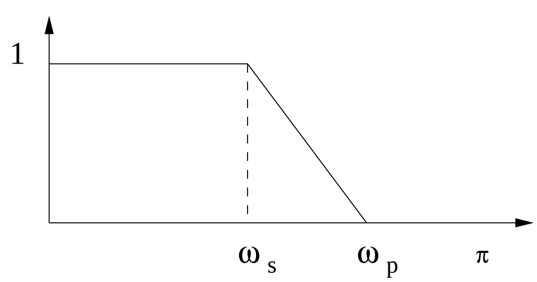

Figure 2: Magnitude response of a lowpass filter

3. OBJECTIVE FUNCTION

We require an expression for the objective function that can be symbollically differentiated to implement the framework. In this section, we derive the objective functions to be used in the minimization problem to measure the deviation from ideal magnitude response, phase response, and quality factors. The final objective function is formed by weighting these various measures.

3.1. Deviation in magnitude response

Since we are optimizing only lowpass filters, the ideal magnitude response is shown in Fig. 2. Given integrable weighting functions $F_p(\omega)$, $F_t(\omega)$, and $F_s(\omega)$, the deviation from the ideal magnitude responses in the passband, transition band, and stopband, respectively, are given by

$$\sigma_p = \int_0^{\omega_p} F_p(\omega)(\mid H(e^{j\omega})\mid -1)^2\, d\omega \tag{5}$$

$$\sigma_t = \int_{\omega_p}^{\omega_s} F_t(\omega)(\mid H(e^{j\omega})\mid - \frac{\omega_s-\omega}{\omega_s-\omega_p})^2\, d\omega \tag{6}$$

$$\sigma_s = \int_{\omega_s}^{\pi} F_s(\omega)\mid H(e^{j\omega})\mid^2\, d\omega \tag{7}$$

3.2. Deviation in the phase response

We could specify a differentiable expression as the desired phase response, and measure the deviation from the desired phase response in the objective function. In this section, we measure the deviation from linear phase over a portion of the passband $[0, \omega_l]$, where $\omega_l \leq \omega_p$. When using (4), the phase response is wrapped, and the denominators of the arctan functions could become zero. To prevent these problems, we apply constraints, as explained in Section 4. The objective function is given by

$$\sigma_{phase} = \int_0^{\omega_l}(\angle H(e^{j\omega}) - m\omega)^2\, d\omega \tag{8}$$

where m is the slope of the best line through the phase response over the desired portion of the passband given by

$$m = \frac{\int_0^{\omega_l}\omega\ \angle H(e^{j\omega})d\omega}{\int_0^{\omega_l}\omega^2 d\omega} \tag{9}$$

We require a differentiable expression for m to use in the SQP framework. Symbolic integration of (9) is not possible. Since the transfer function of a filter with real coefficients is conjugate symmetric, the phase response is an odd function. We approximate the phase response as a four-term Taylor series around $\omega = 0$:

$$\angle \tilde{H}(e^{j\omega}) = w\left(h_1 + h_3\frac{\omega^2}{\omega_l^2} + h_5\frac{\omega^4}{\omega_l^4} + h_7\frac{\omega^6}{\omega_l^6}\right)$$

After substituting $\tilde{H}(e^{j\omega})$ in (9) and solving for $\tilde{m}$,

$$\tilde{m} = h_1 + \frac{3}{5}h_3 + \frac{3}{7}h_5 + \frac{3}{9}h_7 \tag{10}$$

We compute $\tilde{m}$ as the weighted mean of two first-order terms as

$$\tilde{m} = \alpha\frac{\angle \tilde{H}(e^{jr_1\omega_l})}{r_1\omega_l} + \beta\frac{\angle \tilde{H}(e^{jr_2\omega_l})}{r_2\omega_l} \tag{11}$$

where $0 < r_1, r_2 < 1$. We compute α, β, r_1, and r_2 such that for every seventh-order polynomial, $\angle \tilde{H}(e^{j\omega})$, the right-hand side of (11) is identically equal to $\tilde{m}$. Substituting $\tilde{H}(e^{j\omega})$ and $\tilde{m}$ in (11),

$$\begin{aligned} h_1 + \tfrac{3}{5}h_3 + \tfrac{3}{7}h_5 + \tfrac{3}{9}h_7 = \; & (\alpha+\beta)h_1 \\ & +(\alpha r_1^2 + \beta r_2^2)h_3 \\ & +(\alpha r_1^4 + \beta r_2^4)h_5 \\ & +(\alpha r_1^6 + \beta r_2^6)h_7 \end{aligned} \tag{12}$$

We require (12) to hold for all real h_1, h_3, and h_5. We solve the system of equations

$$\begin{aligned} \alpha + \beta &= 1 \\ \alpha r_1^2 + \beta r_2^2 &= 3/5 \\ \alpha r_1^4 + \beta r_2^4 &= 3/7 \\ \alpha r_1^6 + \beta r_2^6 &= 3/9 \end{aligned}$$

This system has multiple solutions. The solution that we implement is

$$\begin{bmatrix} \alpha \\ \beta \\ r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0.416333 \\ 0.583666 \\ 0.538469 \\ 0.906180 \end{bmatrix} \tag{13}$$

Using the above, we compute $\tilde{m}$ from (11) and approximate m as $\tilde{m}$ in (8) to obtain the objective function.

For a zero outside of or on the unit circle, i.e. a zero at $q_l = re^{j\theta}$ for $r \geq 1$, becomes a point of discontinuity in the phase expression in (4) at $\omega = \theta$. At these zero locations, phase is wrapped and hence not differentiable. Since we desire to optimize the phase response in the passband, we want to prevent phase discontinuities in the passband. The passband should not contain zeros; otherwise, the magnitude response will be disrupted. We add constraints to force the zeros to remain outside of the passband to ensure that the phase expression is differentiable and well-behaved in the passband.

3.3. Filter Quality

The quality factor is a measure of the sensitivity of the pole locations. A perturbation in a pole location leads to unexpected oscillations and/or more attenuation in the filter response than designed. The quality factor also reflects the sharpness in the phase and magnitude response. We define the quality factor Q_k of a pole pair $p_k = a_k e^{\pm jb_k}$ as

$$Q_k = \frac{\sqrt{(1+a_k^2)^2 - 4a_k^2 cos^2(b_k)}}{2(1-a_k^2)} \tag{14}$$

where $a_k^2 < 1$ and $0.5 \leq Q < \infty$. Lower quality factors are desirable.

We define a measure of overall filter quality as

$$Q_{\text{eff}} = \left(\prod_{k=1}^{n} Q_k\right)^{\frac{1}{n}} \tag{15}$$

This effective quality factor is the geometric mean of quality factors of the conjugate pole pairs. Since $Q_{\text{eff}} \geq 0.5$, we use $\sigma_q = Q_{\text{eff}} - 0.5$ in the objective measure of deviation in filter quality from the ideal.

3.4. Complete objective function

The overall objective function is expressed as a linear combination or a weighted mean of the various objective measures defined. The total objective is given by σ

$$\sigma = \frac{W_p}{\omega_p}\sigma_p + \frac{W_t}{\omega_s - \omega_p}\sigma_t + \frac{W_s}{\pi - \omega_s}\sigma_s + \frac{W_{phase}}{\omega_l}\sigma_{phase} + W_q\sigma_q \tag{16}$$

where W_p, W_t, W_s, W_{phase} and W_q are positive real weighting factors. The individual terms are normalized by the frequency range so that the values are comparable. The expression for σ is positive and real valued. It is also differentiable under certain conditions which are enforced through constraints as discused in Section 4.

4. CONSTRAINTS

We place constraints on magnitude response, quality factors, numerical stability, and filter stability. The magnitude response constraints occur at uniformly spaced passband frequencies w_i and stopband frequencies w_l:

$$\begin{aligned} 1 - \delta_p \; < \; & |H(e^{j\omega_i})| < 1 + \delta_p \\ & |H(e^{j\omega_l})| < \delta_s \end{aligned}$$

We also add a constraint on the maximum value of the quality factor for the poles. Users can set the maximum value of $Q_{\max}$ for a particular implementation technology. Zero locations are constrained to be outside of the passband so as to avoid phase discontinuities in the passband (see Section 3.2). Poles locations are constrained to be within the unit circle to ensure the stability of the filter.

5. FORMULATION

Mathematica is used to generate MATLAB programs that perform the optimization of a filter with a particular number of pole pairs and zero pairs. The total objective function, the constraints and their derivatives are computed symbolically in the *Mathematica* environment. MATLAB scripts to compute these expressions are also generated by *Mathematica*. The authors of [4] describe the generation of

MATLAB programs from symbolic expressions using *Mathematica*. A driver program is also generated by the symbolic tool. This is used to specify numerical values for the desired filter properties and the weights to use in the objective function. The driver MATLAB script uses the built-in MATLAB SQP function to perform the optimization.

6. EXAMPLE

We use the framework to optimize an elliptic lowpass filter with four zeros and four poles. The magnitude specifications were $\omega_p = 1$ rad, passband ripple of 0.05, $\omega_s = 1.8$ rad, and stopband ripple of 0.01. The pole locations of the initial elliptic filter and the optimized filter are shown in Table 1. In the optimization, we constrained the maximum quality factor to 2, and optimized for linear phase over the entire passband. Figs. 3 and 4 show the magnitude and phase responses of the filters. The phase response of the optimized filter is more linear over the passband and at the same time the quality factors have been reduced. The optimized filter satisfies the magnitude specification.

original filter		optimized filter	
Pole location	Q	Pole location	Q
$0.5176 \pm j0.3264$	0.72	$0.2470 \pm j0.2399$	0.57
$0.4584 \pm j0.7602$	3.62	$0.2815 \pm j0.7355$	2.00

Table 1: Pole pair locations and quality factors of filter before and after optimization

7. CONCLUSION

We develop an extensible SQP-based framework for joint optimization of behavioral and implementation properties of digital IIR filters. The free parameters are the pole-zero locations. We program *Mathematica* to compute the gradients of the objective function and constraints, and convert the SQP problem into MATLAB programs to perform the optimization. The automated approach eliminates algebraic and programming mistakes. We use the framework to design a digital IIR filter with near-linear phase over the passband and minimized quality factors, subject to constraints on the magnitude response. The framework is available at

http://www.ece.utexas.edu/~bevans/
projects/syn_filter_software.html

8. REFERENCES

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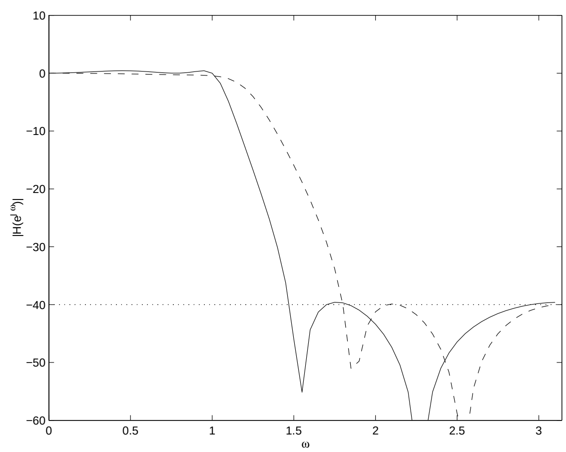

Figure 3: Magnitude response of the initial fourth-order lowpass elliptic filter (solid line) and optimized filter (dashed line). Both filters meet the magnitude specifications of $\omega_p = 1$ rad, $\delta_p = 0.05$, $\omega_s = 1.8$ rad, and $\delta_s = 0.01$.

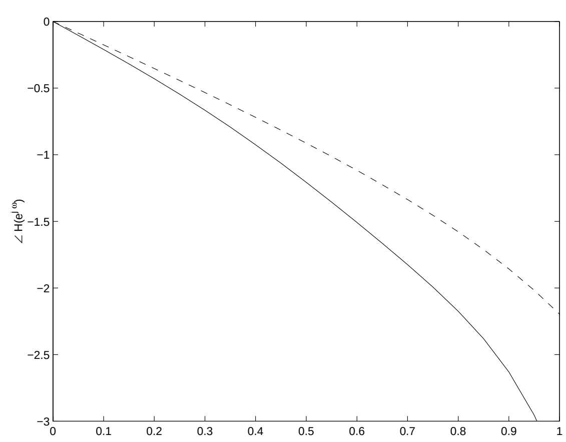

Figure 4: Phase response in the passband of the initial fourth-order lowpass elliptic filter (solid line) and optimized filter (dashed line). In the optimized filter, the phase response is more linear over the passband.