MATRIX GAIN MODEL FOR VECTOR COLOR ERROR DIFFUSION

Niranjan Damera-Venkata and *Brian L. Evans*

Embedded Signal Processing Laboratory
Department of Electrical and Computer Engineering
The University of Texas at Austin, Austin TX 78712-1084 USA
http://signal.ece.utexas.edu/
{damera-v,bevans}@ece.utexas.edu

ABSTRACT

Traditional error diffusion halftoning produces high quality binary images from digital grayscale images. Error diffusion shapes the quantization noise power into the high frequency regions where the human eye is the least sensitive. Error diffusion may be extended to color images by using error filters with matrix-valued coefficients to take into account the correlation among color planes. We propose a new matrix gain model to linearize vector color error diffusion. The model predicts key characteristics of color error diffusion, esp. linear frequency distortion and color noise shaping.

1. INTRODUCTION

Traditional grayscale error diffusion halftoning quantizes an eight bit/pixel grayscale image to a one bit/pixel image for reproduction on binary devices. The reproduction is high quality because error diffusion shapes the quantization noise into the high frequencies (a.k.a. "blue noise") where the human visual system is less sensitive [1]. In addition to adding noise, grayscale error diffusion also sharpens the image [2, 3]. The amount of sharpening depends on the error filter.

Kite, Evans and Bovik [3] quantify the sharpening and noise introduced by grayscale error diffusion by linearizing error diffusion. They replace the quantizer with the "linear gain model" developed by Ardalan and Paulos [4] for sigma-delta modulation. The model accurately predicts the noise shaping and image sharpening in error diffused halftones.

This paper generalizes the linear model of grayscale error diffusion in [3] to vector color error diffusion [5] by replacing the linear gain model with a new matrix gain model and by using properties of filters with matrix-valued coefficients. The new model includes the earlier model [3] as a special case. The new model describes vector color diffusion in the frequency domain, and predicts noise shaping and linear frequency distortion produced by halftoning.

Section 2 introduces the matrix gain model for vector color error diffusion. Section 3 validates the matrix gain model by predicting linear frequency distortion and noise shaping effects of vector color error diffusion. Section 4 concludes the paper by summarizing the results.

This research was supported by an NSF CAREER Award under Grant MIP-9702707 and by a grant from HP Laboratories, Palo Alto, CA.

2. LINEARIZING COLOR ERROR DIFFUSION

Fig. 1 shows a block diagram of vector color error diffusion halftoning. When halftoning red-green-blue (RGB) images, the quantizer output for each color channel at any pixel is exactly one element from the discrete set $\mathcal{O} = \{-1, 1\}$. Here -1 represents a red, green or blue dot, depending on the color channel, whereas 1 represents the absence of a dot for that color channel. We quantize each color channel using a scalar quantizer. The quantizer $\mathbf{Q}(\cdot)$ is defined by

$$\mathbf{Q}(\mathbf{u}) = \begin{pmatrix} Q(u_1) & Q(u_2) & Q(u_3) \end{pmatrix}^T \tag{1}$$

$$Q(u_i) = \begin{cases} 1 & u_i \geq 0 \\ -1 & u_i < 0 \end{cases} \tag{2}$$

where $\mathbf{u}$ is a column vector and u_i, $i = 1, 2, 3$, represents the red, green and blue components of the color vector to be quantized.

The filter in the feedback loop has matrix-valued coefficients. The filter operates on the quantization error sequence $\mathbf{e}(\mathbf{m})$ to produce the feedback signal sequence according to

$$\mathcal{H}\mathbf{e}(\mathbf{m}) = \sum_{\mathbf{k} \in \mathcal{S}} \tilde{\mathbf{h}}(\mathbf{k})\mathbf{e}(\mathbf{m} - \mathbf{k}) \tag{3}$$

where $\mathbf{m}$ and $\mathbf{k}$ are two-dimensional vectors, $\tilde{\mathbf{h}}(\cdot)$ is a 3×3 matrix valued sequence, and $\mathcal{S}$ is the filter support.

We model the quantizer of Fig. 1 by a constant linear transformation denoted by a matrix $\tilde{\mathbf{K}}$ which is applied to the signal components of the quantizer input plus spatially-varying additive noise $\mathbf{n}(\mathbf{m})$ applied to the noise components (components uncorrelated with the input signal) of the quantizer input, as shown in Fig. 2. This is a generalization of modeling the quantizers in sigma-delta modulators [4] and grayscale error diffusion [3, 6]. Correlation among the signal color planes is represented by the off-diagonal terms in the matrix $\tilde{\mathbf{K}}$. We choose the matrix $\tilde{\mathbf{K}}$ to minimize the error in approximating the quantizer with a linear transformation, in the linear minimum mean squared error (LMMSE) sense,

$$\tilde{\mathbf{K}} = \arg\min_{\tilde{\mathbf{A}}} E[\| \mathbf{b}(\mathbf{m}) - \tilde{\mathbf{A}}\mathbf{u}(\mathbf{m}) \|^2] \tag{4}$$

where $\mathbf{b}(\cdot)$ represents the quantizer output process, and $\mathbf{u}(\cdot)$ represents the quantizer input process. The solution to (4) when $\mathbf{b}(\cdot)$ and $\mathbf{u}(\cdot)$ are wide sense stationary processes is

$$\tilde{\mathbf{K}} = \tilde{\mathbf{C}}_{\mathbf{bu}} \tilde{\mathbf{C}}_{\mathbf{uu}}^{-1} \quad (5)$$

where $\tilde{\mathbf{C}}_{\mathbf{bu}}$ and $\tilde{\mathbf{C}}_{\mathbf{uu}}$ are covariance matrices [7]. As a direct consequence of this modeling [7], the noise process $\mathbf{n}(\cdot)$ due to the signal approximation error is uncorrelated with the signal input to the quantizer $\mathbf{u}(\cdot)$. We will analyze error diffusion, assuming a matrix gain of $\tilde{\mathbf{K}}$ for the signal path and a matrix gain of $\tilde{\mathbf{I}}$ (identity matrix) for the noise path. This corresponds to using the estimator to estimate signal components in the output of the quantizer from signal components at its input, and assuming an uncorrelated noise injection to model the noise. In this way, one may treat the signal shaping and noise shaping independently.

Analyzing the linearized vector color error diffusion model shown in Fig. 2 using z-transforms yields

$$Z\{\mathcal{H}\mathbf{e}(\mathbf{m})\} = \tilde{\mathbf{H}}(\mathbf{z})\mathbf{E}(\mathbf{z}) \quad (6)$$

By analyzing the signal path and ignoring the noise path,

$$\mathbf{X}(\mathbf{z}) = \mathbf{U}(\mathbf{z}) + \tilde{\mathbf{H}}(\mathbf{z})\mathbf{E}(\mathbf{z}) \quad (7)$$

$$\mathbf{E}(\mathbf{z}) = (\tilde{\mathbf{K}} - \tilde{\mathbf{I}})\mathbf{U}(\mathbf{z}) \quad (8)$$

$$\mathbf{B}_s(\mathbf{z}) = \tilde{\mathbf{K}}\mathbf{U}(\mathbf{z}) \quad (9)$$

By manipulating (7), (8), and (9), the response to the signal component becomes

$$\mathbf{B}_s(\mathbf{z}) = \tilde{\mathbf{K}}[\tilde{\mathbf{I}} + \tilde{\mathbf{H}}(\mathbf{z})(\tilde{\mathbf{K}} - \tilde{\mathbf{I}})]^{-1}\mathbf{X}(\mathbf{z}) \quad (10)$$

By considering the contribution of the noise component $\mathbf{B}(\mathbf{z})$ to the output $\mathbf{B}_n(\mathbf{z})$,

$$\mathbf{B}_n(\mathbf{z}) = \mathbf{N}(\mathbf{z}) + \mathbf{U}(\mathbf{z}) \quad (11)$$

$$\mathbf{U}(\mathbf{z}) = -\tilde{\mathbf{H}}(\mathbf{z})\mathbf{E}(\mathbf{z}) \quad (12)$$

$$\mathbf{E}(\mathbf{z}) = \mathbf{N}(\mathbf{z}) \quad (13)$$

By rearranging (11), (12) and (13),

$$\mathbf{B}_n(\mathbf{z}) = [\tilde{\mathbf{I}} - \tilde{\mathbf{H}}(\mathbf{z})]\mathbf{N}(\mathbf{z}) \quad (14)$$

The overall system response is given by

$$\mathbf{B}(\mathbf{z}) = \mathbf{B}_s(\mathbf{z}) + \mathbf{B}_n(\mathbf{z}) \quad (15)$$

Equations (10) and (14) reduce to the analogous ones for grayscale error diffusion [3], in which the error filter coefficients and signal gain are scalar valued.

3. VALIDATING THE MATRIX GAIN MODEL

In this section, we validate the matrix gain model by using it to predict the linear distortion and noise shaping effects of vector color error diffusion. Sections 3.1 and 3.2 show that the signal path distortion given by (10) accurately models the linear distortion that the original color image is subjected to in vector color error diffusion. In Section 3.3, we validate that the model accurately predicts the noise shaping behavior of vector color error diffusion.

3.1. Validation by linearly distorting the original

We linearly distort the original image without introducing quantization noise by processing the original image of Fig. 5(a) by using (10). This is equivalent to processing the original image according to Fig. 2, with the additive noise ignored. Fig. 5(b) shows the resulting image. Fig. 5(c) shows the result of halftoning with a fixed error filter. Fixed error filters for the validation process were obtained by terminating the adaptive algorithm in [5] after a random number of iterations. Figs. 5(b) and 5(c) have comparable linear distortion. To see this, we simply form the residual image by subtracting Fig. 5(b) from Fig. 5(c). The result is shown in Fig. 5(d). The residual in Fig. 5(d) is uncorrelated with the original and represents quantization noise. This is consistent with the modeling of Section 2. To quantify the degree of correlation of the residual image with the original image, we introduce a correlation matrix defined by

$$\left[\tilde{\mathbf{C}}_{\mathbf{rx}}\right]_{ij} = \rho_{r_i x_j}$$

where $\rho_{r_i x_j}$ represents the correlation coefficient [7] between the color plane i in the residual and the color plane j in the original image. The correlation matrix for the residual shown in Fig. 5(d), with respect to the original image shown in Fig. 5(a) is

$$\tilde{\mathbf{C}}_{\mathbf{rx}} = \begin{pmatrix} 0.0067 & 0.0007 & 0.0051 \\ 0.0065 & 0.0039 & 0.0049 \\ 0.0082 & 0.0040 & 0.0062 \end{pmatrix}$$

3.2. Validation by creating an undistorted halftone

The model predicts that the linear distortion suffered by the color input image is given by (10). This means that if one prefilters the input color image by using the filter

$$\tilde{\mathbf{G}}(\mathbf{z}) = [\tilde{\mathbf{I}} + \tilde{\mathbf{H}}(\mathbf{z})(\tilde{\mathbf{K}} - \tilde{\mathbf{I}})]\tilde{\mathbf{K}}^{-1} \quad (16)$$

then the resulting halftone should exhibit a flat low-frequency response with respect to the original color image. Fig. 3 shows error diffusion modified to include the prefilter. Fig. 3 is exactly equivalent to Fig. 4 when $\tilde{\mathbf{L}} = \tilde{\mathbf{L}}_{opt} = \tilde{\mathbf{K}}^{-1} - \tilde{\mathbf{I}}$, whenever $[\tilde{\mathbf{I}} - \tilde{\mathbf{H}}(\mathbf{z})]$ is invertible [8].

For grayscale error diffusion, this result reduces to the result derived in [3] in which the gain is scalar-valued and the error filter has scalar coefficients. Fig. 4 feeds a linear transformation $\tilde{\mathbf{L}}$ of the input image into the quantizer input. The matrix gain model predicts that the linear distortion in the halftoning process must be eliminated. To check this result, we first compute the residual of an unmodified halftone (i.e. halftoned using $\tilde{\mathbf{L}} = \tilde{\mathbf{0}}$) with respect to the original. Fig. 5(a) shows the original image to be halftoned. Fig. 5(c) shows the halftone image, which was halftoned with $\tilde{\mathbf{L}} = \tilde{\mathbf{0}}$ (usual vector color error diffusion). Fig. 6(a) shows the residual with respect to the original by subtracting Fig. 5(c) from Fig. 5(a). The correlation matrix for the residual is

$$C_{\mathbf{rx}} = \begin{pmatrix} 0.3204 & 0.2989 & 0.0999 \\ 0.2787 & 0.3295 & 0.1605 \\ 0.2063 & 0.2952 & 0.1836 \end{pmatrix}$$

Fig. 6(b) shows the halftone image, which was halftoned with $\tilde{\mathbf{L}} = \tilde{\mathbf{K}}^{-1} - \tilde{\mathbf{I}}$ (modified vector color error diffusion). Fig. 6(c) shows the residual with respect to the original by subtracting Fig. 6(b) from Fig. 5(a). The correlation matrix for the residual is

$$\tilde{C}_{\mathbf{rx}} = \begin{pmatrix} 0.0052 & 0.0009 & 0.0040 \\ 0.0054 & 0.0023 & 0.0020 \\ 0.0058 & 0.0011 & 0.0027 \end{pmatrix}$$

This shows that the linear distortion has been removed by modified vector color error diffusion, since the residual with respect to the original is uncorrelated noise (signal components in the residual have been eliminated).

Knox [2] shows that the error image for grayscale error diffusion $\mathbf{e}(\mathbf{m})$ is correlated with the input image. Knox also shows that the sharpness of halftones increases as the correlation of the error image with the input increases. Kite, Evans and Bovik [3] show that by adding dither, the quantization error may be decorrelated with respect to the input, and the sharpening (linear distortion) effects of error diffusion vanish. They also conclude [3] that image sharpening is due to the fact that the input to the error filter contains signal components, which are fed back and shaped. Since the system has a highpass response, this results in the halftone being sharper than the original image.

We will show by using the matrix gain model that in the case of modified error diffusion (Fig. 4), halftoning with the value of $\tilde{\mathbf{L}}$ which cancels linear distortion, is a sufficient condition for the error image (input to the error filter) to be free of signal components from the input image.

By replacing the quantizer in Fig. 4 with a gain matrix $\tilde{\mathbf{K}}$ and analyzing the signal path,

$$\begin{aligned} \mathbf{E}_s(\mathbf{z}) &= \tilde{\mathbf{K}}\left[\tilde{\mathbf{L}}\mathbf{X}(\mathbf{z}) + \mathbf{U}(\mathbf{z})\right] - \mathbf{U}(\mathbf{z}) \\ &= \tilde{\mathbf{K}}\,\tilde{\mathbf{L}}\,\mathbf{X}(\mathbf{z}) + \left[\tilde{\mathbf{K}} - \tilde{\mathbf{I}}\right]\mathbf{U}(\mathbf{z}) \end{aligned} \tag{17}$$

Since

$$\mathbf{U}(\mathbf{z}) = \mathbf{X}(\mathbf{z}) - \tilde{\mathbf{H}}(\mathbf{z})\mathbf{E}_s(\mathbf{z}) \tag{18}$$

we obtain

$$\left[\tilde{\mathbf{I}} + \left(\tilde{\mathbf{K}} - \tilde{\mathbf{I}}\right)\tilde{\mathbf{H}}(\mathbf{z})\right]\mathbf{E}_s(\mathbf{z}) = \left[\tilde{\mathbf{K}}\tilde{\mathbf{L}} + \tilde{\mathbf{K}} - \tilde{\mathbf{I}}\right]\mathbf{X}(\mathbf{z}) \tag{19}$$

By substituting $\tilde{\mathbf{L}} = \tilde{\mathbf{K}}^{-1} - \tilde{\mathbf{L}}$ into (19), $\mathbf{E}_s(\mathbf{z}) = 0$. Hence, there are no signal components in the error image. To check this prediction, and hence validate our modeling, we halftone test images with $\tilde{\mathbf{L}}$ set to cancel linear distortion. Fig. 5(a) shows the original image to be halftoned. Fig. 6(b) shows the halftone image, halftoned with $\tilde{\mathbf{L}} = \tilde{\mathbf{K}}^{-1} - \tilde{\mathbf{I}}$ (modified vector color error diffusion). Fig. 6(d) shows the error image. The correlation matrix for the error image with respect to the original is

$$\tilde{C}_{\mathbf{ex}} = \begin{pmatrix} 0.0455 & 0.0235 & 0.0122 \\ 0.0493 & 0.0144 & 0.0164 \\ 0.0428 & 0.0142 & 0.0150 \end{pmatrix}$$

The low correlation of the error image was predicted by the theory and therefore strongly corroborates it.

3.3. Validation of the noise response

According to our model, the noise shaping is predicted by (14). To verify the prediction, we first compute a residual as described in Section 3.1. This residual is shaped noise. We need to verify that the noise shaping is in fact given by (14). We halftone test images using the optimal distortion cancelling method described in Section 3.2. This corresponds to halftoning with the value of $\tilde{\mathbf{L}} = \tilde{\mathbf{K}}^{-1} - \tilde{\mathbf{L}}$. The matrix gain model predicts that the input to the error filter has no signal components. The input to the error filter in this case is $\mathbf{N}(\mathbf{z})$. We then filter this noise image (i.e. input to the error filter) according to (14), to form a predicted residual. If the noise shaping equation is correct, then this residual must be spectrally close to the actual residual image. This was indeed found to be the case. Fig. 7 shows radially averaged spectra of the three color planes of the actual residual noise image and the residual computed using the noise shaping predicted from the model. The close agreement of the spectra confirms the predictions of the matrix gain model.

4. CONCLUSION

We present a framework for the analysis of vector color error diffusion systems that use matrix-valued error filter coefficients. For the multi-channel quantizer, we introduce a matrix gain model. The model linearizes color error diffusion, which permits the use of linear systems theory to derive the signal and noise transfer functions. We validated the model by predicting the linear frequency distortion and color noise shaping effects of color error diffusion.

5. REFERENCES

[1] R. Ulichney, *Digital Halftoning.* Cambridge, MA: MIT Press, 1987.

[2] K. Knox, "Error image in error diffusion," *Proc. SPIE Image Processing Algorithms and Techniques III*, vol. 1657, pp. 268–279, Feb. 1992.

[3] T. D. Kite, B. L. Evans, and A. C. Bovik, "Modeling and quality assessment of halftoning by error diffusion," *IEEE Trans. Image Processing*, vol. 9, pp. 909–922, May 2000.

[4] S. Ardalan and J. Paulos, "An analysis of nonlinear behavior in delta-sigma modulators," *IEEE Trans. Circuits and Systems*, vol. 34, pp. 593–603, June 1987.

[5] L. Akarun, Y. Yardimici, and A. E. Cetin, "Adaptive methods for dithering color images," *IEEE Trans. Image Processing*, vol. 6, pp. 950–955, July 1997.

[6] T. D. Kite, B. L. Evans, A. C. Bovik, and T. L. Sculley, "Digital halftoning as 2-D delta-sigma modulation," *Proc. IEEE Conf. Image Processing*, vol. 1, pp. 799–802, Oct. 1997.

[7] H. Stark and J. W. Woods, *Probability, Random Processes and Estimation Theory for Engineers.* Englewood Cliffs, NJ: Prentice-Hall, 1986.

[8] N. Damera-Venkata, *Analysis and Design of Vector Error Diffusion Systems for Image Halftoning.* PhD thesis, Dept. of Electrical and Comp. Eng., The University of Texas at Austin, Austin, TX, Dec. 2000.

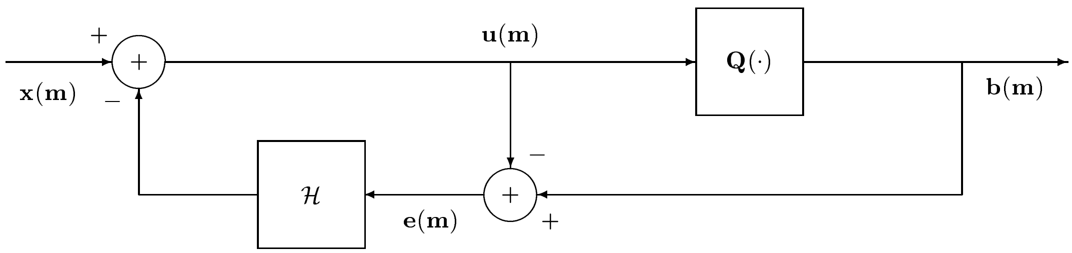


Figure 1: System block diagrams for vector color error diffusion halftoning where $\mathcal{H}$ represents a fixed 2-D nonseparable FIR error filter with matrix valued coefficients.

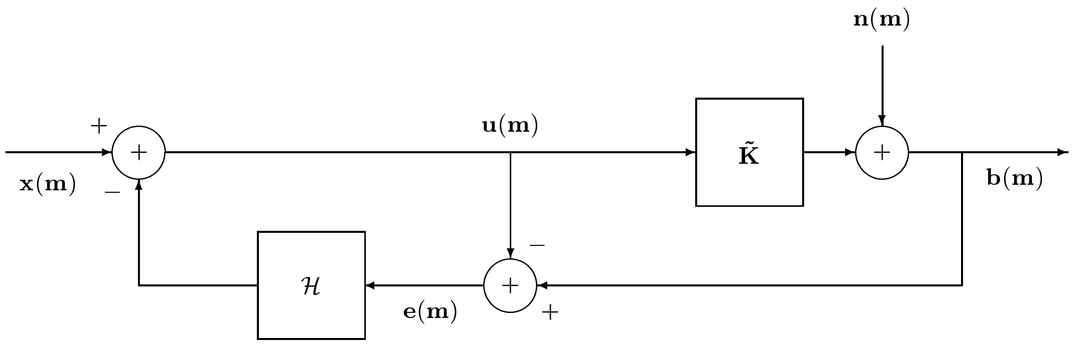


Figure 2: System block diagram for vector color error diffusion model, where $\tilde{\mathbf{K}}$ represents a linear transformation of the signal component of $\mathbf{u}(\mathbf{m})$ and $\mathbf{n}(\mathbf{m})$ is a noise process uncorrelated with the signal component of $\mathbf{u}(\mathbf{m})$.

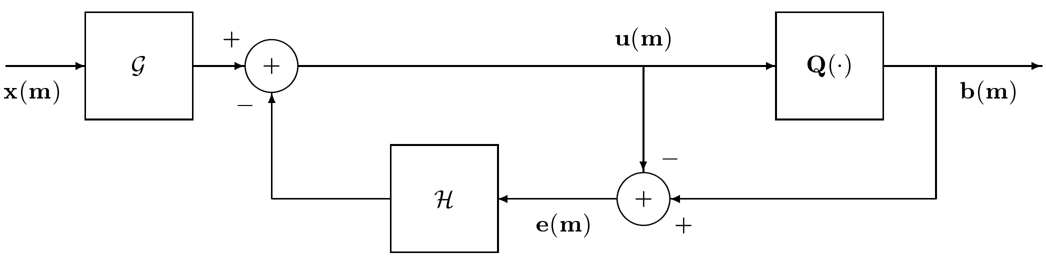


Figure 3: System block diagrams for vector color error diffusion halftoning with a fixed pre-filter $\mathcal{G}$ having matrix valued coefficients.

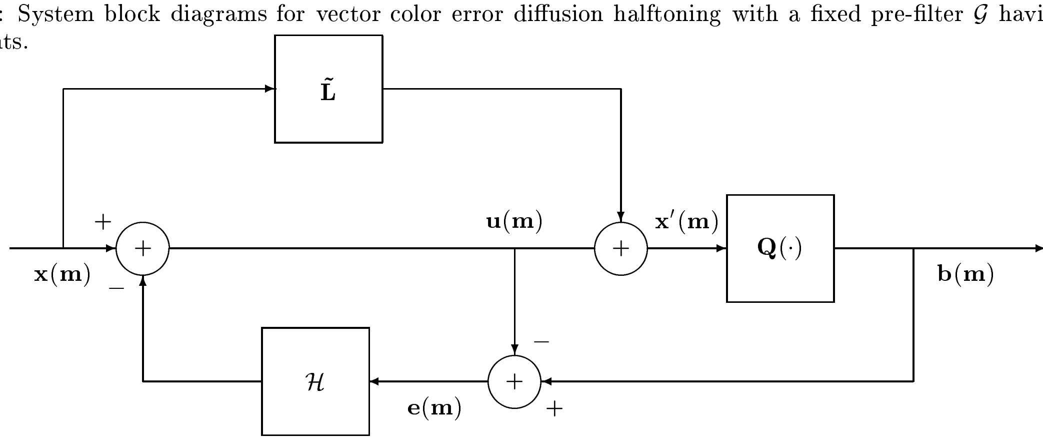
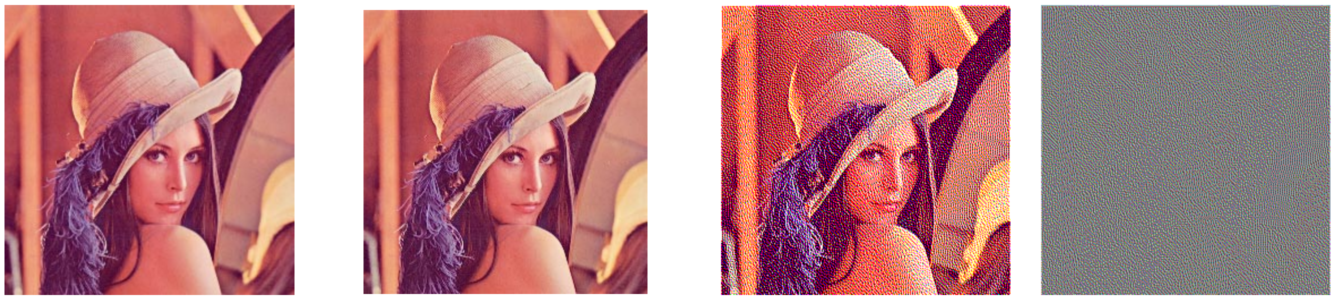
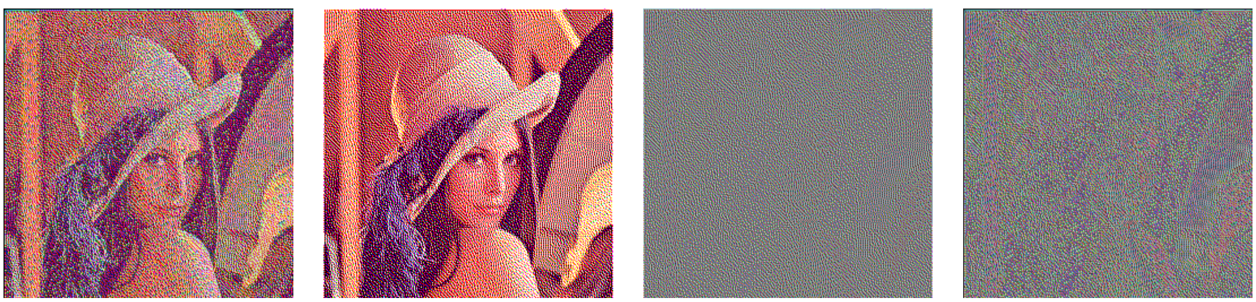


Figure 4: System block diagrams for modified vector color error diffusion halftoning. $\tilde{\mathbf{L}}$ represents a constant linear transformation.



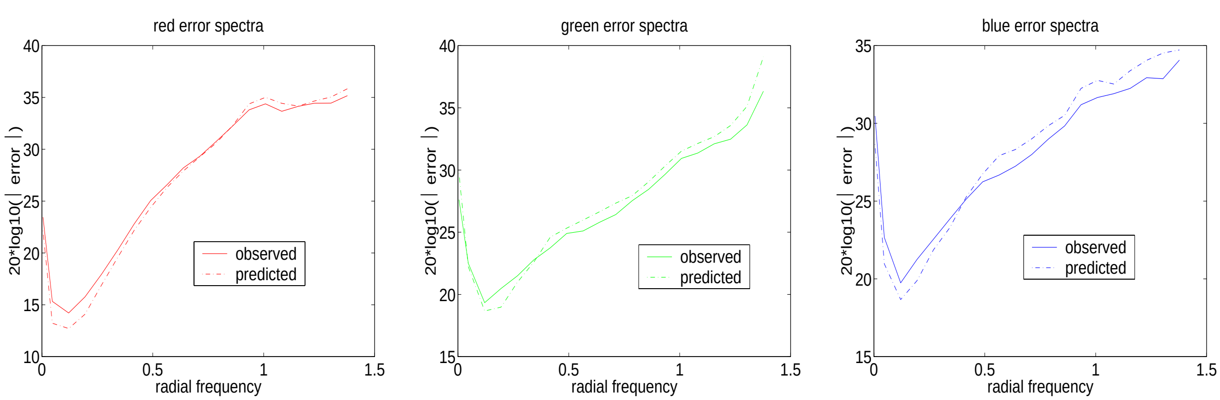
(a) 256 $\times$ 256 *lenna* (b) *lena* generated using model $\tilde{\mathbf{K}}$ (c) Halftoned *lenna* (d) Residual noise image

Figure 5: Validation of matrix gain model by linearly distorting the original image. Here the residual image has been scaled using a full-scale contrast stretch for display purposes.



(a) Residual when $\tilde{\mathbf{L}} = \tilde{\mathbf{0}}$ (b) Halftone using $\tilde{\mathbf{L}}_{opt}$ (c) Residual using $\tilde{\mathbf{L}}_{opt}$ (d) Error image using $\tilde{\mathbf{L}}_{opt}$

Figure 6: Validation of matrix gain model by creating an undistorted halftone. Here the residual image and the input to the error filter have been scaled using a full-scale contrast stretch for display purposes.



(a) (b) (c)

Figure 7: Predicted and actual radially averaged spectra for residual noise image: (a) red, (b) green and (c) blue planes. Solid lines indicate actual spectra while the dashed lines represent predicted spectra