16. Consider the system shown in Fig. 4.37(a).

(a) What is the system type? Compute the steady-state tracking error due to a ramp input \( r(t) = r_o t_1(t) \).

(b) For the modified system shown in Fig. 47(b), give the value of \( H_f \) so the system is type 2 for reference inputs and compute the \( K_a \) in this case.

(c) Is the resulting type 2 property of this system robust with respect to changes in \( H_f \)? i.e., will the system remain type 2 if \( H_f \) changes slightly?

\[ E(s) = [1 - T(s)]R(s) \]
\[ = \left[ \frac{1}{1 + G(s)} \right] R(s) \]
\[ = \frac{s(\tau s + 1)}{s(\tau s + 1) + A s^2} \]

The steady-state tracking error using the FVT (assuming stability) is
\[ e_{ss} = \lim_{s \to 0} sE(s) = \frac{r_o}{A} \]
(b)

\[ Y(s) = \frac{A}{s(\tau s + 1)} U(s) \]

\[ U(s) = H_f s R(s) + H_r R(s) - Y(s) \]

\[ Y(s) = \frac{A(H_f s + H_r)}{s(\tau s + 1) + A} R(s) \]

The tracking error is,

\[ E(s) = R(s) - Y(s) \]

\[ = \frac{s(\tau s + 1) + A - A(H_f s + H_r)}{s(\tau s + 1) + A} R(s) \]

\[ = \frac{\tau s^2 + (1 - AH_f)s + A(1 - H_r)}{s(\tau s + 1) + A} \]

To get zero steady-state error with respect to a ramp, the numerator in the above equation must have a factor \( s^2 \). For this to happen, let

\[ H_r = 1 \]
\[ AH_f = 1 \]

Then

\[ E(s) = \frac{\tau s^2}{s(\tau s + 1) + A} R(s) \]

and, with \( R(s) = \frac{r_0}{s^2} \), apply the FVT (assuming stability) to obtain

\[ e_{ss} = 0. \]

Thus the system will be type 2 with \( K_a = \frac{\tau}{A} \).

(c) No, the system is not robust type 2 because the property is lost if either \( H_r \) or \( H_f \) changes slightly.
27. Consider the system shown in Fig. 4.46.

(a) Find the transfer function from the reference input to the tracking error.

(b) For this system to respond to inputs of the form \( r(t) = t^n1(t) \) (where \( n < q \)) with zero steady-state error, what constraint is placed on the open-loop poles \( p_1, p_2, \ldots, p_q \)?

![Figure 4.46: Control system for Problem 27](image)

Solution:

(a)

\[
\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} = \frac{\prod_{i=1}^{q} (s + p_i)}{\prod_{i=1}^{q} (s + p_i) + 1}
\]

(b)

\[ r(t) = t^n \implies R(s) = \frac{n!}{s^{n+1}} \]

\[
e_{ss} = \lim_{s \to 0} s^{n} \frac{n!}{s^{n+1}} \frac{\prod_{i=1}^{q} (s + p_i)}{\prod_{i=1}^{q} (s + p_i) + 1}
\]

If \( e_{ss} \) is to be zero the system must have at least \( n + 1 \) poles at the origin:

\[
e_{ss} = \lim_{s \to 0} s^{n} \frac{n!}{s^{n+1}} \frac{s^{n+1} \prod_{i=1}^{q} (s + p_i)}{s^{n+1} \prod_{i=1}^{q} (s + p_i) + 1} = 0
\]
33. For a system with impulse response \( h(t) \), prove that the velocity constant is given by

\[
\frac{1}{K_v} = \int_{0}^{\infty} t h(t) \, dt,
\]

and the acceleration constant is given by

\[
\frac{1}{K_a} = -\frac{1}{2} \int_{0}^{\infty} t^2 h(t) \, dt.
\]

**Solution:**

If we define

\[
Y(s) = \frac{K_p}{1 + K_p} R(s) - \frac{1}{K_v} s R(s) - \frac{1}{K_a} s^2 R(s) - \ldots
\]

If \( r(t) \) is a unit impulse, then \( y(t) \) is the impulse response \( h(t) \). Since \(-\frac{1}{K_v}\) is the zero-frequency derivative of \( Y(s) \), then from the Laplace relationship,

\[
H(s) = \int_{0}^{\infty} h(t) e^{-st} \, dt
\]

Differentiation with respect to \( s \) yields,

\[
H'(s) = \int_{0}^{\infty} -th(t) e^{-st} \, dt
\]

Substituting \( s = 0 \) in the above equation we find,

\[
\frac{1}{K_v} = \int_{0}^{\infty} th(t) \, dt
\]

Differentiating yields

\[
H^{(2)}(s) = \int_{0}^{\infty} t^2 h(t) e^{-st} \, dt
\]

But since \(-\frac{1}{K_a}\) is \( \frac{1}{2} H^{(2)}(0) \) we obtain

\[
\frac{1}{K_a} = -\frac{1}{2} \int_{0}^{\infty} t^2 h(t) \, dt.
\]
Problems and Solutions for Section 6.3

18. (a) Sketch the Nyquist plot for an open-loop system with transfer function $1/s^2$; that is, sketch

$$\frac{1}{s^2}|_{s=C_1},$$

where $C_1$ is a contour enclosing the entire RHP, as shown in Fig. 6.17. (Hint: Assume $C_1$ takes a small detour around the poles at $s = 0$, as shown in Fig. 6.27.)

(b) Repeat part (a) for an open-loop system whose transfer function is $G(s) = 1/(s^2 + \omega_0^2)$.

Solution:

(a) $G(s) = \frac{1}{s^2}

Note that the portion of the Nyquist diagram on the right side below that corresponds to the bode plot is from B' to C'. The large loop from F' to A' to B' arises from the detour around the 2 poles at the origin.

(b) $G(s) = \frac{1}{s^2 + \omega_0^2}$

Note here that the portion of the Nyquist plot coming directly from a Bode plot is the portion from A' to E'. That portion includes a 180° arc that arose because of the detour around the pole on the
imaginary axis.
19. Sketch the Nyquist plot based on the Bode plots for each of the following systems, then compare your result with that obtained using the MATLAB command `nyquist`:

(a) \( KG(s) = \frac{K(s + 2)}{s + 10} \)

(b) \( KG(s) = \frac{K}{(s + 10)(s + 2)^2} \)

(c) \( KG(s) = \frac{K(s + 10)(s + 1)}{(s + 100)(s + 2)^3} \)

(d) Using your plots, estimate the range of \( K \) for which each system is stable, and qualitatively verify your result using a rough sketch of a root-locus plot.

**Solution**:

(a) 

(b) 

(c) 

(d)
\[ N = 0, \ P = 0 \implies Z = N + P = 0 \]

The closed-loop system is stable for any \( K > 0 \).

(b) The Bode plot shows an initial phase of 0° hence the Nyquist starts on the positive real axis at A’. The Bode ends with a phase of -270° hence the Nyquist ends the bottom loop by approaching the origin from the positive imaginary axis (or an angle of -270°).

The magnitude of the Nyquist plot as it crosses the negative real axis is 0.00174. It will not encircle the \(-1/K\) point until \( K = 1/0.00174 = 576 \).

i. \( 0 < K < 576 \)

\[ N = 0, \ P = 0 \implies Z = N + P = 0 \]

The closed-loop system is stable.

ii. \( K > 576 \)

\[ N = 2, \ P = 0 \implies Z = N + P = 2 \]

The closed-loop system has two unstable roots as verified by the root locus.
(c) The Bode plot shows an initial phase of $0^\circ$ hence the Nyquist starts on the positive real axis at $A'$. The Bode ends with a phase of $-180^\circ$ hence the Nyquist ends the bottom loop by approaching the origin from the negative real axis (or an angle of $-180^\circ$).

It will never encircle the $-1/K$ point, hence it is always stable. The root locus below confirms that.

\[ N = 0, P = 0 \implies Z = N + P = 0 \]

The closed-loop system is stable for any $K > 0$. 
20. Draw a Nyquist plot for

\[ KG(s) = \frac{K(s + 1)}{s(s + 3)} \]  

choosing the contour to be to the right of the singularity on the \( j\omega \)-axis. and determine the range of \( K \) for which the system is stable using the Nyquist Criterion. Then redo the Nyquist plot, this time choosing the contour to be to the left of the singularity on the imaginary axis and again check the range of \( K \) for which the system is stable using the Nyquist Criterion. Are the answers the same? Should they be?

\textbf{Solution :}

If you choose the contour to the right of the singularity on the origin, the Nyquist plot looks like this:

From the Nyquist plot, the range of \( K \) for stability is \( -\frac{1}{3} < 0 \) \((N = 0, P = 0 \implies Z = N + P = 0)\). So the system is stable for \( K > 0 \).

Similarly, in the case with the contour to the left of the singularity on the origin, the Nyquist plot is:
From the Nyquist plot, the range of $K$ for stability is $\frac{1}{K} < 0 (N = -1, P = 1 \implies Z = N + P = 0)$. So the system is stable for $K > 0$.

The way of choosing the contour around singularity on the $j\omega$-axis does not affect its stability criterion. The results should be the same in either way. However, it is somewhat less cumbersome to pick the contour to the right of a pole on the imaginary axis so that there are no unstable poles within the contour, hence $P=0$. 
21. Draw the Nyquist plot for the system in Fig. 6.90. Using the Nyquist stability criterion, determine the range of $K$ for which the system is stable. Consider both positive and negative values of $K$.

**Figure 6.90: Control system for Problem 21**

![Control system diagram](image)

**Solution:**

The characteristic equation:

\[
1 + K \frac{1}{s^2 + 2s + 2} \frac{1}{s + 1} = 0
\]

\[
G(s) = \frac{1}{(s + 1)(s^2 + 2s + 2)}
\]

For positive $K$, note that the magnitude of the Nyquist plot as it crosses the negative real axis is 0.1, hence $K < 10$ for stability. For negative $K$, the entire Nyquist plot is essentially flipped about the imaginary axis, thus the magnitude where it crosses the negative real axis will be 0.5 and the stability limit is that $|K| < 2$. Therefore, the range of $K$ for stability is $-2 < K < 10$. 
Problems and Solutions for Section 6.4

23. The Nyquist plot for some actual control systems resembles the one shown in Fig. 6.91. What are the gain and phase margin(s) for the system of Fig. 6.91 given that \( \alpha = 0.4 \), \( \beta = 1.3 \), and \( \phi = 40^\circ \). Describe what happens to the stability of the system as the gain goes from zero to a very large value. Sketch what the corresponding root locus must look like for such a system. Also sketch what the corresponding Bode plots would look like for the system.

Figure 6.91: Nyquist plot for Problem 23

Solution:

The phase margin is defined as in Figure 6.33, \( PM = \phi (\omega = \omega^*) \), but now there are several gain margins! If the system gain is increased (multiplied) by \( \frac{1}{|\alpha|} \) or decreased (divided) by \( |\beta| \), then the system will go unstable. This is a conditionally stable system. See Figure 6.39 for a typical root locus of a conditionally stable system.

\[
\begin{align*}
\text{gain margin} &= -20 \log |\alpha|_{dB} (\omega = \omega_H) \\
\text{gain margin} &= +20 \log |\beta|_{dB} (\omega = \omega_L)
\end{align*}
\]

For a conditionally stable type of system as in Fig. 6.39, the Bode phase plot crosses \(-180^\circ\) twice; however, for this problem we see from the
Nyquist plot that it crosses 3 times! For very low values of gain, the entire Nyquist plot would be shrunk, and the -1 point would occur to the left of the negative real axis crossing at $\omega_o$, so there would be no encirclements and the system would be stable. As the gain increases, the -1 point occurs between $\omega_o$ and $\omega_L$ so there is an encirclement and the system is unstable. Further increase of the gain causes the -1 point to occur between $\omega_L$ and $\omega_H$ (as shown in Fig. 6.91) so there is no encirclement and the system is stable. Even more increase in the gain would cause the -1 point to occur between $\omega_H$ and the origin where there is an encirclement and the system is unstable. The root locus would look like Fig. 6.39 except that the very low gain portion of the loci would start in the LHP before they loop out into the RHP as in Fig. 6.39. The Bode plot would be vaguely like that drawn below: