Determinant maximization with linear matrix inequality constraints

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Abstract

The problem of maximizing the determinant of a matrix subject to linear matrix inequalities arises in many fields, including computational geometry, statistics, system identification, experiment design, and information and communication theory. It can also be considered as a generalization of the semidefinite programming problem.

We give an overview of the applications of the determinant maximization problem, pointing out simple cases where specialized algorithms or analytical solutions are known. We then describe an interior-point method, with a simplified analysis of the worst-case complexity and numerical results that indicate that the method is very efficient, both in theory and in practice. Compared to existing specialized algorithms (where they are available), the interior-point method will generally be slower; the advantage is that it handles a much wider variety of problems.

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- Associated software will be available at URL `http://www-isl.stanford.edu/people/boyd` and from anonymous FTP to `isl.stanford.edu` in `pub/boyd/maxdet`. This will include an implementation in Matlab and C, with a user interface (matlab, sdpsol).
- Future versions of the paper will also be made available at the same URL and FTP-site (`pub/boyd/reports/maxdet.ps.Z`).

1 Introduction

We consider the optimization problem

$$\begin{array}{ll} \text{minimize} & c^T x + \log\det G(x)^{-1} \\ \text{subject to} & G(x) > 0 \\ & F(x) \geq 0, \end{array} \tag{1}$$

where the optimization variable is the vector $x \in \mathbf{R}^m$. The functions $G : \mathbf{R}^m \to \mathbf{R}^{l\times l}$ and $F : \mathbf{R}^m \to \mathbf{R}^{n\times n}$ are affine:

$$\begin{aligned} G(x) &= G_0 + x_1 G_1 + \cdots + x_m G_m, \\ F(x) &= F_0 + x_1 F_1 + \cdots + x_m F_m, \end{aligned}$$

where $G_i = G_i^T$ and $F_i = F_i^T$. The inequality signs in (1) denote matrix inequalitiesΓ*i.e.*Γ $G(x) > 0$ means $z^T G(x) z > 0$ for all nonzero z and $F(x) \geq 0$ means $z^T F(x) z \geq 0$ for all z. We call $G(x) > 0$ and $F(x) \geq 0$ (strict and nonstrictΓrespectively) *linear matrix inequalities* (LMIs) in the variable x. We will refer to problem (1) as a maxdet-problemΓsince in many cases the term $c^T x$ is absentΓso the problem reduces to maximizing the determinant of $G(x)$ subject to LMI constraints.

The maxdet-problem is a convex optimization problemΓ*i.e.*Γthe objective function $c^T x + \log\det G(x)^{-1}$ is convex (on $\{x \mid G(x) > 0\}$)Γ and the constraint set is convex. IndeedΓ LMI constraints can represent many common convex constraintsΓ including linear inequalitiesΓconvex quadratic inequalitiesΓand matrix norm and eigenvalue constraints (see Alizadeh[Ali95]ΓBoydΓEl GhaouiΓFeron and Balakrishnan[BEFB94]ΓLewis and Overton[LO96]Γ Nesterov and Nemirovsky[NN94Γ§6.4]Γand Vandenberghe and Boyd[VB96]).

The maxdet-problem (1) can be solved by several algorithms for general convex programming that are efficient in theory (*i.e.*Γworst case complexity)Γ*e.g.*Γthe ellipsoid method (Yudin and Nemirovsky[YN77]ΓShor[Sho77]). It can also be solved by general nonlinear programming methodsΓprovided they are modified to handle the (nonsmooth) LMI constraints. In this paper we describe an interior-point method that solves the maxdet-problem very efficientlyΓboth in worst-case complexity theory and in practice. The method we describe shares many features of interior-point methods for linear and semidefinite programming. In particularΓour computational experience (which is limited to problems of moderate size — several hundred variablesΓ with matrices up to 100×100) indicates that the method we describe solves the maxdet-problem (1) in a number of iterations that hardly varies with problem sizeΓand typically ranges between 5 and 50; each iteration involves solving a system of linear equations.

Maxdet-problems arise in many fieldsΓincluding computational geometryΓstatisticsΓand information and communication theoryΓso the duality theory and algorithms we develop have wide application. In some of these applicationsΓand for very simple forms of the problemΓthe maxdet-problems can be solved by specialized algorithms orΓin some casesΓanalytically. Our interior-point algorithm will generally be *slower* than the specialized algorithms (when the specialized algorithms can be used). The *advantage* of our approach is that it is much more general; it handles a much wider variety of problems. The analytical solutions or specialized

algorithms, for example, cannot handle the addition of (convex) constraints; our algorithm for general maxdet-problems does.

In the remainder of §1, we describe some interesting special cases of the maxdet-problem, such as semidefinite programming and analytic centering. In §2 we describe examples and applications of maxdet-problems, pointing out analytical solutions where they are known, and interesting extensions that can be handled as general maxdet-problems. In §3, we describe a duality theory for maxdet-problems, pointing out connections to semidefinite programming duality. Our interior-point method for solving the maxdet-problem (1) is developed in §4–§9. We describe two variations: a simple 'short-step' method, for which we can prove polynomial worst-case complexity, and a 'long-step' or adaptive step predictor-corrector method which has the same worst-case complexity, but is much more efficient in practice. We finish with some numerical experiments. The appendix contains key proofs and formulas.

Let us now describe some special cases of the maxdet-problem.

Semidefinite programming

When $G(x) = 1$, the maxdet-problem reduces to

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) \geq 0, \end{array} \tag{2}$$

which is known as a *semidefinite program* (SDP). Semidefinite programming unifies a wide variety of convex optimization problems, *e.g.*, linear programming,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

which can be expressed as an SDP with $F(x) = \mathbf{diag}(b - Ax)$. For surveys of the theory and applications of semidefinite programming, see [Ali95], [BEFB94], [Lew96], [LO96], [NN94, §6.4], and [VB96].

Analytic centering

When $c = 0$ and $F(x) = 1$, the maxdet-problem (1) reduces to

$$\begin{array}{ll} \text{minimize} & \log\det G(x)^{-1} \\ \text{subject to} & G(x) > 0, \end{array} \tag{3}$$

which we call the *analytic centering* problem. We will assume that the feasible set $\mathbf{X} = \{x \mid G(x) > 0\}$ is nonempty and bounded, which implies that the matrices G_i, $i = 1, \ldots, m$, are linearly independent, and that the objective $\phi(x) = \log\det G(x)^{-1}$ is strictly convex on $\mathbf{X}$ (see, *e.g.*, [VB96] or [BE93]). Since the objective function grows without bound as x approaches the boundary of $\mathbf{X}$, there is a unique solution $x^\star$ of (3). We call $x^\star$ the *analytic center* of the LMI $G(x) > 0$. The analytic center of an LMI generalizes the analytic center of a set of linear inequalities, introduced by Sonnevend [Son86, Son91].

Since the constraint cannot be active at the analytic center$\Gamma x^\star$ is characterized by the optimality condition $\nabla\phi(x^\star) = 0$:

$$(\nabla\phi(x^\star))_i = -\mathbf{Tr} G_i G(x^\star)^{-1} = 0, \quad i = 1, \ldots, m \tag{4}$$

(see for example Boyd and El Ghaoui[BE93]).

The analytic center of an LMI is important for several reasons. We will see in §5 that the analytic center can be computed very efficientlyΓso it can be used as an easily computed robust solution of the LMI. Analytic centering also plays an important role in interior-point methods for solving the more general maxdet-problem (1). Roughly speaking$\Gamma$the interior-point methods solve the general problem by solving a sequence of analytic centering problems.

Parametrization of LMI feasible set

Let us restore the term $c^T x$ to the analytic centering problem:

$$\begin{array}{ll} \text{minimize} & c^T x + \log\det G(x)^{-1} \\ \text{subject to} & G(x) > 0, \end{array} \tag{5}$$

retaining our assumption that $\mathbf{X}$ is nonempty and boundedΓso the matrices G_i are linearly independent and the objective function is strictly convex. ThusΓproblem (5) has a unique solution $x^\star(c)\Gamma$which satisfies the optimality conditions $c + \nabla\phi(x^\star(c)) = 0\Gamma i.e.\Gamma$

$$\mathbf{Tr} G_i G(x^\star(c))^{-1} = c_i, \quad i = 1, \ldots, m.$$

Thus for each $c \in \mathbf{R}^m\Gamma$we have a (readily computed) point $x^\star(c)$ in the feasible set $\mathbf{X}$.

ConverselyΓgiven a point $x \in \mathbf{X}\Gamma$define $c \in \mathbf{R}^m$ by $c_i = \mathbf{Tr} G(x)^{-1} G_i \Gamma i = 1, \ldots, m$. Evidently we have $x = x^\star(c)$. In other wordsΓthere is a one-to-one correspondence between vectors $c \in \mathbf{R}^m$ and vectors $x \in \mathbf{X}$: the mapping $c \mapsto x^\star(c)$ is a parametrization of the feasible set $\mathbf{X}$ of the strict LMI $G(x) > 0\Gamma$with parameter $c \in \mathbf{R}^m$.

This parametrization of $\mathbf{X}$ is related to the *Legendre transform* of the convex function $\log\det G(x)^{-1}\Gamma$defined by

$$\mathcal{L}(y) = -\inf\{-y^T x + \log\det G(x)^{-1} \mid G(x) > 0\}.$$

Maximal lower bounds in the positive definite cone

Here we consider a simple example of the maxdet-problem. Let $A_i = A_i^T \Gamma i = 1, \ldots, L\Gamma$be positive definite matrices in $\mathbf{R}^{p\times p}$. A matrix X is a lower bound of the matrices A_i if $X \le A_i\Gamma$ $i = 1, \ldots, L$; it is a maximal lower bound if there is no lower bound Y with $Y \ne X\Gamma Y \ge X$.

Since the function $\log\det X^{-1}$ is monotone decreasing with respect to the positive semidefinite cone$\Gamma i.e.\Gamma$

$$0 < X \le Y \Longrightarrow \log\det Y^{-1} \le \log\det X^{-1},$$

we can compute a maximal lower bound A_{mlb} by solving

$$\begin{array}{ll} \text{minimize} & \log\det X^{-1} \\ \text{subject to} & X > 0 \\ & X \le A_i, \quad i = 1, \ldots, L. \end{array} \tag{6}$$

This is a maxdet-problem with $p(p+1)/2$ variables (the elements of the matrix X). The constraints are the strict LMI $X > 0$Γand L nonstrict LMIs $A_i - X \geq 0$Γwhich we can also consider as diagonal blocks of one single block diagonal LMI

$$\mathbf{diag}(A_1 - X, A_2 - X, \ldots, A_L - X) \geq 0.$$

Of course there are other maximal lower bounds; replacing $\log\det X^{-1}$ by any other monotone decreasing matrix functionΓ$e.g.$Γ$-\mathbf{Tr}X$ or $\mathbf{Tr}X^{-1}$Γwill also yield (other) maximal lower bounds. The maximal lower bound A_{mlb} obtained by solving (6)ΓhoweverΓhas the property that it is invariant under congruence transformationsΓ$i.e.$Γif the matrices A_i are transformed to TA_iT^TΓwhere $T \in \mathbf{R}^{p\times p}$ is nonsingularΓthen the maximal lower bound obtained from (6) is $TA_{\mathrm{mlb}}T^T$.

2 Examples and applications

In this section we catalog examples and applications. The reader interested only in duality theory and solution methods for the maxdet-problem can skip directly to §3.

2.1 Minimum volume ellipsoid containing given points

Perhaps the earliest and best known application of the maxdet-problem arises in the problem of determining the minimum volume ellipsoid that contains given points x^1Γ...Γx^K in $\mathbf{R}^n$ (orΓequivalentlyΓtheir convex hull $\mathbf{Co}\{x^1, \ldots, x^K\}$). This problem has applications in cluster analysis (Rosen[Ros65]ΓBarnes[Bar82])Γand robust statistics (in ellipsoidal peeling methods for outlier detection; see Rousseeuw and Leroy[RL87Γ§7]).

We describe the ellipsoid as $\mathcal{E} = \{x \mid \|Ax+b\| \leq 1\}$Γwhere $A = A^T > 0$Γso the volume of $\mathcal{E}$ is proportional to $\det A^{-1}$. Hence the minimum volume ellipsoid that contains the points x^i can be computed by solving the convex problem

$$\begin{array}{ll} \text{minimize} & \log\det A^{-1} \\ \text{subject to} & \|Ax^i + b\| \leq 1, \ i = 1, \ldots, K \\ & A = A^T > 0, \end{array} \tag{7}$$

where the variables are $A = A^T \in \mathbf{R}^{n\times n}$ and $b \in \mathbf{R}^n$. The norm constraints $\|Ax^i + b\| \leq 1$Γ which are just convex quadratic inequalities in the variables A and bΓcan be expressed as LMIs

$$\begin{bmatrix} I & Ax^i + b \\ (Ax^i + b)^T & 1 \end{bmatrix} \geq 0.$$

These LMIs can in turn be expressed as one large block diagonal LMIΓso (7) is a maxdet-problem in the variables A and b.

Minimum volume ellipsoid containing ellipsoids

There are many interesting variations and extensions of this problem. As an exampleΓconsider the problem of finding the minimum volume ellipsoid $\mathcal{E}_0$ containing K given ellipsoids

Figure 1: Minimum volume ellipsoid containing five given ellipsoids. Finding such an ellipsoid can be cast as a maxdet-problem, hence efficiently solved.

$\mathcal{E}_1, \ldots, \mathcal{E}_K$. For this problem we describe the ellipsoids as sublevel sets of convex quadratic functions:

$$\mathcal{E}_i = \{x \mid x^T A_i x + 2b_i^T x + c_i \leq 0\}, \quad i = 0, \ldots, K.$$

The solution can be found by solving the following maxdet-problem in the variables $A_0 = A_0^T$ $b_0$and K scalar variables τ_i:

$$\begin{array}{ll} \text{minimize} & \log\det A_0^{-1} \\ \text{subject to} & A_0 = A_0^T > 0 \\ & \tau_1 \geq 0, \ldots, \tau_K \geq 0 \\ & \begin{bmatrix} A_0 & b_0 & 0 \\ b_0^T & -1 & b_0^T \\ 0 & b_0 & -A_0 \end{bmatrix} - \tau_i \begin{bmatrix} A_i & b_i & 0 \\ b_i^T & c_i & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq 0, \quad i = 1, \ldots, K. \end{array}$$

(c_0 is given by $c_0 = b_0^T A_0^{-1} b_0 - 1$.) See [BEFB94p.43] for details. Figure 1 shows an instance of the problem.

The ellipsoid of least volume containing a set is often called the Löwner ellipsoid (after Danzer Grünbaum and Klee[DGK63 p.139]) or the Löwner-John ellipsoid (Grötschel Lovász and Schrijver[GLS88 p.69]). John in [Joh85] has shown that if one shrinks the minimum volume outer ellipsoid of a convex set $C \subset \mathbf{R}^n$ by a factor n about its center one obtains an ellipsoid contained in C. Thus the Löwner-John ellipsoid serves as an ellipsoidal approximation of a convex set with bounds that depend only on the ambient dimension and not in any other way on the set C.

2.2 Maximum volume ellipsoid in a polyhedron

A related problem is that of computing the maximum volume ellipsoid inside a polyhedron described by linear inequalitiesΓ

$$\mathcal{P} = \{x \mid a_i^T x \leq b_i,\ i = 1, \ldots, L\}.$$

Here we represent the ellipsoid as $\mathcal{E} = \{Cy + d \mid \|y\| \leq 1\}\Gamma$where we assume$\Gamma$without loss of generality$\Gamma C = C^T > 0$. The volume of $\mathcal{E}$ is proportional to $\det C\Gamma$so the problem can be expressed as

$$\begin{array}{ll} \text{maximize} & \log\det C \\ \text{subject to} & C = C^T > 0 \\ & \mathcal{E} \subseteq \mathcal{P}. \end{array} \tag{8}$$

The constraint $\mathcal{E} \subseteq \mathcal{P}$ can be replaced by a set of LMIs

$$\begin{aligned} \mathcal{E} \subseteq \mathcal{P} \quad &\Longleftrightarrow \quad \sup_{x \in \mathcal{E}} a_i^T x \leq b_i,\ i = 1, \ldots, L, \\ &\Longleftrightarrow \quad \sup_{\|y\| \leq 1} a_i^T C y + a_i^T d \leq b_i,\ i = 1, \ldots, L, \\ &\Longleftrightarrow \quad \|C a_i\| + a_i^T d \leq b_i,\ \ i = 1, \ldots, L, \\ &\Longleftrightarrow \quad \begin{bmatrix} (b_i - a_i^T d) I & C a_i \\ a_i^T C & b_i - a_i^T d \end{bmatrix} \geq 0,\ \ i = 1, \ldots, L, \end{aligned}$$

so problem (9) can be cast as the maxdet-problem

$$\begin{array}{ll} \text{maximize} & \log\det C \\ \text{subject to} & C = C^T > 0 \\ & \begin{bmatrix} (b_i - a_i^T d) I & C a_i \\ a_i^T C & b_i - a_i^T d \end{bmatrix} \geq 0,\ \ i = 1, \ldots, L, \end{array} \tag{9}$$

with variables $C = C^T \in \mathbf{R}^{n \times n}$ and $d \in \mathbf{R}^n$.

Nesterov and Nemirovsky[NN94Γ§6.5]$\Gamma$and Khachiyan and Todd[KT93] describe interior-point algorithms for computing the maximum-volume ellipsoid in a polyhedron described by linear inequalities (as well as the minimum-volume ellipsoid covering a polytope described by its vertices).

Many other geometrical problems involving ellipsoidal approximations can be formulated as maxdet-problems. References [BEFB94Γ§3.7] and [Che80] give several examplesΓincluding the maximum volume ellipsoid contained in the intersection or in the sum of given ellipsoidsΓ and the minimum volume ellipsoid containing the sum of given ellipsoids. For other ellipsoidal approximation problemsΓsuboptimal solutions can be computed via maxdet-problems.

Ellipsoidal approximations of convex sets are used in control theory and signal processing in *bounded-noise* or *set-membership* techniques. These techniques were first introduced for state estimation (seeΓ *e.g.*Γ Schweppe[Sch68Γ Sch73]Γ Witsenhausen[Wit68]Γ Bertsekas and Rhodes[BR71]ΓChernousko[Che80ΓChe94])$\Gamma$and later applied to system identification (Fogel and Huang[Fog79ΓFH82]ΓNorton [Nor86ΓNor87Γ§8.6]ΓWalter and Piet-Lahanier[WPL90]Γ

CheungΓYurkovich and Passino[CYP93])Γand signal processing (Deller [Del89]. For a survey emphasizing signal processing applicationsΓsee Deller et al. [DNO93]).

Other applications include the *method of inscribed ellipsoids* developed by TarasovΓ KhachiyanΓand Erlikh[TKE88]Γand design centering (Sapatnekar[Sap92]).

2.3 Maximum volume rectangle in a polyhedron

Let $\mathcal{P} = \{x \in \mathbf{R}^n \mid Ax \leq b\}$ be a polyhedron in $\mathbf{R}^n$Γwhere $A \in \mathbf{R}^{m\times n}$. We consider the problem of inscribing the rectangle $\mathcal{R}(\underline{x}, \overline{x}) = \{x \mid \underline{x} \leq x \leq \overline{x}\}$ of maximum volume in $\mathcal{P}$. (The inequalities $\underline{x} \leq x \leq \overline{x}$ are componentwise inequalities: $\underline{x} \leq x$ means $\underline{x}_i > x_i$Γ $i = 1, \ldots, n$.) The optimization variables are the two vectors $\underline{x}$ and $\overline{x}$.

The volume of $\mathcal{R}(\underline{x}, \overline{x})$ is equal to $\prod_i(\overline{x}_i - \underline{x}_i)$Γso the problem can be expressed as

$$\begin{array}{ll} \text{maximize} & \prod_{i=1}^{n}(\overline{x}_i - \underline{x}_i) \\ \text{subject to} & \mathcal{R}(\underline{x}, \overline{x}) \subseteq \mathcal{P} \\ & \underline{x} < \overline{x}. \end{array}$$

The constraint $\mathcal{R}(\underline{x}, \overline{x}) \subseteq \mathcal{P}$ can be cast as a set of m inequalities

$$A^+\overline{x} - A^-\underline{x} \leq b,$$

where $A^+_{ij} = \max\{0, A_{ij}\}$ and $A^-_{ij} = \max\{0, -A_{ij}\}$. Therefore we can compute the maximum volume rectangle enclosed in the polytope by solving

$$\begin{array}{ll} \text{maximize} & \prod_{i=1}^{n}(\overline{x}_i - \underline{x}_i) \\ \text{subject to} & A^+\overline{x} - A^-\underline{x} \leq b \\ & \underline{x} < \overline{x}. \end{array} \tag{10}$$

This can be cast as a maxdet-problemΓwith variable $x = \begin{bmatrix}\underline{x}^T & \overline{x}^T\end{bmatrix}^T$Γand $G(x) = \mathbf{diag}(\overline{x} - \underline{x})$.

This problem can be extended in many ways. For exampleΓwe can require that the rectangle contains L given points y^iΓ$i = 1, \ldots, L$Γby adding the linear inequalities $\underline{x} \leq y^i \leq \overline{x}$.

2.4 Matrix completion problems

Positive definite matrix completion

In a positive definite matrix completion problem we are given a symmetric matrix $A_f \in \mathbf{R}^{n\times n}$Γ some entries of which are fixed; the remaining entries are to be chosen so that the resulting matrix is positive definite.

Let the positions of the free (unspecified) entries be given by the index pairs (i_k, j_k)Γ (j_k, i_k)Γ$k = 1, \ldots, m$. We can assume that the diagonal elements are fixedΓ$i.e.$Γ$i_k \neq j_k$ for all k. (If a diagonal elementΓsay the (l, l)thΓis freeΓwe take it to be very largeΓwhich makes

the lth row and column of A_f irrelevant.) The positive definite completion problem can be cast as an SDP feasibility problem:

$$\begin{array}{ll} \text{find} & x \in \mathbf{R}^m \\ \text{such that} & A(x) \triangleq A_f + \sum_{k=1}^{m} x_k \left(E_{i_k j_k} + E_{j_k i_k}\right) > 0, \end{array}$$

where E_{ij} denotes the matrix with all elements zero except the (i,j) elementΓwhich is equal to one. Note that the set $\mathbf{X} = \{x \mid A(x) > 0\}$ is bounded since the diagonal elements of $A(x)$ are fixed.

Maximum entropy completion

The analytic center of the LMI $A(x) > 0$ is sometimes called the *maximum entropy completion* of A_f. From the optimality conditions (4)Γwe see that the maximum entropy completion $x^\star$ satisfies

$$2\mathbf{Tr} E_{i_k j_k} A(x^\star)^{-1} = 2\left(A(x^\star)^{-1}\right)_{i_k j_k} = 0,\ k = 1, \ldots, m,$$

*i.e.*Γthe matrix $A(x^*)^{-1}$ has a zero entry in every location corresponding to an unspecified entry in the original matrix. This is a very useful property in many applications; seeΓfor exampleΓDempster[Dem72]Γor Dewilde and Ning[DN90].

Parametrization of all positive definite completions

As an extension of the maximum entropy completion problemΓconsider

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr} C A(x) + \log\det A(x)^{-1} \\ \text{subject to} & A(x) > 0, \end{array} \tag{11}$$

where $C = C^T$ is given. This problem is of the form (5); the optimality conditions are

$$A(x^*) > 0, \quad \left(A(x^*)^{-1}\right)_{i_k j_k} = C_{i_k j_k},\ \ k = 1, \ldots, m, \tag{12}$$

*i.e.*Γthe inverse of the optimal completion matches the given matrix C in every free entry. IndeedΓthis gives a parametrization of all positive definite completions: a positive definite completion $A(x)$ is uniquely characterized by specifying the elements of its inverse in the free locationsΓ *i.e.*Γ $(A(x)^{-1})_{i_k j_k}$. Problem (11) has been studied by Bakonyi and Woerdeman[BW95].

Contractive completion

A related problem is the contractive completion problem: given a (possibly nonsymmetric) matrix A_f and m index pairs (i_k, j_k)Γ$k = 1, \ldots, m$Γfind a matrix

$$A(x) = A_f + \sum_{k=1}^{m} x_k E_{i_k, j_k}.$$

with spectral norm (maximum singular value) less than one.

This can be cast as a semidefinite programming feasibility problem [VB96]: find x such that

$$\begin{bmatrix} I & A(x) \\ A(x)^T & I \end{bmatrix} > 0. \tag{13}$$

One can define a maximum entropy solution as the solution that maximizes the determinant of (13)Γ*i.e.*Γsolves the maxdet-problem

$$\begin{array}{ll} \text{maximize} & \log\det(I - A(x)^T A(x)) \\ \text{subject to} & \begin{bmatrix} I & A(x) \\ A(x)^T & I \end{bmatrix} > 0. \end{array} \tag{14}$$

See Nævdal and Woerdeman[NW92]ΓHelton and Woerdeman[HW93]. For a statistical interpretation of (14)Γsee §2.5.

Specialized algorithms and references

Very efficient algorithms have been developed for certain specialized types of completion problems. A well known example is the maximum entropy completion of a positive definite banded Toeplitz matrix (Dym and Gohberg[DG81]ΓDewilde and Deprettere[DD88]). DavisΓ KahanΓand Weinberger[DKW82] discuss an analytic solution for a contractive completion problem with a special (block matrix) form. The methods discussed in this paper solve the *general* problem efficientlyΓalthough they are slower than the specialized algorithms where they are applicable. Moreover they have the advantage that other convex constraintsΓ*e.g.*Γ upper and lower bounds on certain entriesΓare readily incorporated.

Completion problemsΓand specialized algorithms for computing completionsΓhave been discussed by many authorsΓ seeΓ*e.g.*Γ Dym and Gohberg[DG81]Γ GroneΓ JohnsonΓ Sá and Wolkowicz[GJSW84]ΓBarrettΓJohnson and Lundquist[BJL89]ΓLundquist and Johnson[LJ91]Γ Dewilde and Deprettere[DD88]ΓDemboΓMallowsΓand Shepp[DMS89]. Johnson gives a survey in [Joh90]. An interior-point method for an approximate completion problem is discussed in JohnsonΓKroschelΓand Wolkowicz[JKW95].

We refer to Boyd *et al.* [BEFB94Γ§3.5]Γand El Ghaoui[El 96]Γfor further discussion and additional references.

2.5 Risk-averse linear estimation

Let $y = Ax + w$ with $w \sim N(0, I)$ and $A \in \mathbf{R}^{q\times p}$. Here x is an unknown quantity that we wish to estimateΓy is the measurementΓand w is the measurement noise. We assume that $p \leq q$ and that A has full column rank.

A linear estimator $\hat{x} = My$Γwith $M \in \mathbf{R}^{p\times q}$Γis unbiased if $\mathbf{E}\hat{x} = x$Γ*i.e.*Γ$MA = I$. The minimum-variance unbiased estimator is the unbiased estimator that minimizes the error variance

$$\mathbf{E}\|My - x\|^2 = \mathbf{Tr} MM^T = \sum_{i=1}^{p} \sigma_i^2(M),$$

where $\sigma_i(M)$ is the ith largest singular value of M. It is given by $M = A^\dagger$ where $A^\dagger = (A^TA)^{-1}A^T$ is the pseudo-inverse of A. In fact the minimum-variance estimator is optimal in a stronger sense: it not only minimizes $\sum_i \sigma_i^2(M)$ but each singular value $\sigma_i(M)$ separately:

$$MA = I \Longrightarrow \sigma_i(A^\dagger) \leq \sigma_i(M), \quad i = 1, \ldots, p. \tag{15}$$

In some applications estimation errors larger than the mean value are more costly for less desirable than errors less than the mean value. To capture this idea of *risk aversion* we can consider the objective or cost function

$$2\gamma^2 \log \mathbf{E} \exp\left(\frac{1}{2\gamma^2}\|My - x\|^2\right) \tag{16}$$

where the parameter γ is called the *risk-sensitivity parameter*. This cost function was introduced by Whittle in the more sophisticated setting of stochastic optimal control; see [Whi82 §19]. Note that as $\gamma \to \infty$ the risk-sensitive cost (16) converges to the cost $\mathbf{E}\|My - x\|^2$ and is always larger (by convexity of exp and Jensen's inequality). We can gain further insight from the first terms of the series expansion in $1/\gamma^2$:

$$\begin{aligned} 2\gamma^2 \log \mathbf{E} \exp\left(\frac{1}{2\gamma^2}\|\widehat{x} - x\|^2\right) &\simeq \mathbf{E}\|\widehat{x} - x\|^2 + \frac{1}{4\gamma^2}\left(\mathbf{E}\|\widehat{x} - x\|^4 - \left(\mathbf{E}\|\widehat{x} - x\|^2\right)^2\right) \\ &= \mathbf{E}z + \frac{1}{4\gamma^2}\mathrm{var}\ z, \end{aligned}$$

where $z = \|\widehat{x} - x\|^2$ is the squared error. Thus for large γ the risk-averse cost (16) augments the mean-square error with a term proportional to the variance of the squared error.

The unbiased risk averse optimal estimator can be found by solving

$$\begin{array}{ll} \text{minimize} & 2\gamma^2 \log \mathbf{E} \exp\left(\frac{1}{2\gamma^2}\|My - x\|^2\right) \\ \text{subject to} & MA = I, \end{array}$$

which can be expressed as a maxdet-problem. The objective function can be written as

$$\begin{aligned} 2\gamma^2 \log \mathbf{E} \exp\left(\frac{1}{2\gamma^2}\|My - x\|^2\right) &= 2\gamma^2 \log \mathbf{E} \exp\left(\frac{1}{2\gamma^2} w^T M^T M w\right) \\ &= \begin{cases} 2\gamma^2 \log \det(I - (1/\gamma^2)M^TM)^{-1/2} & \text{if } M^TM < \gamma^2 I \\ \infty & \text{otherwise} \end{cases} \\ &= \begin{cases} \gamma^2 \log \det \begin{bmatrix} I & \gamma^{-1}M^T \\ \gamma^{-1}M & I \end{bmatrix}^{-1} & \text{if } \begin{bmatrix} I & \gamma^{-1}M^T \\ \gamma^{-1}M & I \end{bmatrix} > 0 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

so the unbiased risk averse optimal estimator solves the maxdet-problem

$$\begin{array}{ll} \text{minimize} & \gamma^2 \log \det \begin{bmatrix} I & \gamma^{-1}M^T \\ \gamma^{-1}M & I \end{bmatrix}^{-1} \\ \text{subject to} & \begin{bmatrix} I & \gamma^{-1}M^T \\ \gamma^{-1}M & I \end{bmatrix} > 0 \\ & MA = I. \end{array} \tag{17}$$

This is in fact an analytic centering problem, and has a simple analytic solution: the least squares estimator $M = A^\dagger$. To see this we express the objective in terms of the singular values of M:

$$\gamma^2 \log\det \begin{bmatrix} I & \gamma^{-1}M^T \\ \gamma^{-1}M & I \end{bmatrix}^{-1} = \begin{cases} -\gamma^2 \sum_{i=1}^{p} \log(1-\sigma_i^2(M)/\gamma^2)^{-1} & \text{if } \sigma_1(M) < \gamma \\ \infty & \text{otherwise.} \end{cases}$$

It follows from property (15) that the solution is $M = A^\dagger$ if $\|A^\dagger\| < \gamma$, and that the problem is infeasible otherwise. (Whittle refers to the infeasible case, in which the risk-averse cost is always infinite, as 'neurotic breakdown'.)

In the simple case discussed above, the optimal risk-averse and the minimum-variance estimators coincide (so there is certainly no advantage in a maxdet-problem formulation). When additional convex constraints on the matrix M are added, *e.g.*, a given sparsity pattern, or triangular or Toeplitz structure, the optimal risk-averse estimator can be found by including these constraints in the maxdet-problem (17) (and will not, in general, coincide with the minimum-variance estimator).

2.6 Experiment design

Optimal experiment design

As in the previous section, we consider the problem of estimating a vector x from a measurement $y = Ax + w$, where $w \sim N(0, I)$ is measurement noise. The error covariance of the minimum-variance estimator is equal to $A^\dagger(A^\dagger)^T = (A^TA)^{-1}$. We suppose that the rows of the matrix $A = [a_1 \ \ldots \ a_q]^T$ can be chosen among M possible test vectors $v^{(i)} \in \mathbf{R}^p$, $i = 1, \ldots, M$:

$$a_i \in \{v^{(1)}, \ldots, v^{(M)}\}, \quad i = 1, \ldots, q.$$

The goal of experiment design is to choose the vectors a_i so that the error covariance $(A^TA)^{-1}$ is 'small'. We can interpret each component of y as the result of an experiment or measurement that can be chosen from a fixed menu of possible experiments; our job is to find a set of measurements that (together) are maximally informative.

We can write $A^TA = q\sum_{i=1}^{M} \lambda_i v^{(i)} v^{(i)^T}$, where λ_i is the fraction of rows a_k equal to the vector $v^{(i)}$. We ignore the fact that the numbers λ_i are integer multiples of $1/q$, and instead treat them as continuous variables, which is justified in practice when q is large. (Alternatively, we can imagine that we are designing a random experiment: each experiment a_i has the form $v^{(k)}$ with probability λ_k.)

Many different criteria for measuring the size of the matrix $(A^TA)^{-1}$ have been proposed. For example, in E-optimal design, we minimize the norm of the error covariance, $\lambda_{\max}((A^TA)^{-1})$, which is equivalent to maximizing the smallest eigenvalue of A^TA. This is

readily cast as the SDP

$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & \sum_{i=1}^{M} \lambda_i v^{(i)} v^{(i)T} \geq tI \\ & \sum_{i=1}^{M} \lambda_i = 1 \\ & \lambda_i \geq 0, \quad i = 1, \ldots, M \end{array}$$

in the variables $\lambda_1, \ldots, \lambda_M$, and t. Another criterion is A-optimality, in which we minimize $\mathbf{Tr}(A^T A)^{-1}$. This can be cast as an SDP:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{p} t_i \\ \text{subject to} & \begin{bmatrix} \sum_{i=1}^{M} \lambda_i v^{(i)} v^{(i)T} & e_i \\ e_i^T & t_i \end{bmatrix} \geq 0, \quad i = 1, \ldots, p, \\ & \lambda_i \geq 0, \quad i = 1, \ldots, M, \\ & \sum_{i=1}^{M} \lambda_i = 1, \end{array}$$

where e_i is the ith unit vector in $\mathbf{R}^p$, and the variables are λ_i, $i = 1, \ldots, M$, and t_i, $i = 1, \ldots, p$.

In D-optimal design, we minimize the determinant of the error covariance $(A^T A)^{-1}$, which leads to the maxdet-problem

$$\begin{array}{ll} \text{minimize} & \log\det \left(\sum_{i=1}^{M} \lambda_i v^{(i)} v^{(i)T} \right)^{-1} \\ \text{subject to} & \lambda_i \geq 0, \; i = 1, \ldots, M \\ & \sum_{i=1}^{M} \lambda_i = 1. \end{array} \tag{18}$$

In §3 we will derive an interesting geometrical interpretation of the D-optimal matrix A, and show that $A^T A$ determines the minimum volume ellipsoid, centered at the origin, that contains $v^{(1)}, \ldots, v^{(M)}$.

Fedorov[Fed71], Atkinson and Donev[AD92], and Pukelsheim[Puk93] give surveys and additional references on optimal experiment design.

Extensions of D-optimal experiment design

The formulation of D-optimal design as an maxdet-problem has the advantage that one can easily incorporate additional useful convex constraints. For example, one can add linear inequalities $c_i^T \lambda \leq \alpha_i$, which can reflect bounds on the total cost of, or time required to carry out, the experiments.

We can also consider the case where each experiment yields several measurements, *i.e.*, the vectors a_i and $v^{(k)}$ become matrices. The maxdet-problem formulation (18) remains the

Figure 2: A D-optimal experiment design involving 50 test vectors in $\mathbf{R}^2$, with and without the 90-10 constraint. The circle is the origin; the dots are the test vectors that are not used in the experiment (*i.e.*, have a weight $\lambda_i = 0$); the crosses are the test vectors that are used (*i.e.*, have a weight $\lambda_i > 0$). Without the 90-10 constraint, the optimal design allocates all meaurements to only two test vectors. With the constraint, the measurements are spread over ten vectors, with no more than 90% of the measurements allocated to any group of five vectors. See also Figure 3.

same, except that the terms $v^{(k)}v^{(k)T}$ can now have rank larger than one. This extension is useful in conjunction with additional linear inequalities representing limits on cost or time: we can model discounts or time savings associated with performing groups of measurements simultaneously. Suppose, for example, that the cost of simultaneously making measurements $v^{(1)}$ and $v^{(2)}$ is less than the sum of the costs of making them separately. We can take $v^{(3)}$ to be the matrix

$$v^{(3)} = \begin{bmatrix} v^{(1)} & v^{(2)} \end{bmatrix}$$

and assign costs c_1, c_2, and c_3 associated with making the first measurement alone, the second measurement alone, and the two simultaneously, respectively.

Let us describe in more detail another useful additional constraint that can be imposed: that no more than a certain fraction of the total number of experiments, say 90%, is concentrated in less than a given fraction, say 10%, of the possible measurements. Thus we require

$$\sum_{i=1}^{\lfloor 0.1M \rfloor} \lambda_{[i]} \leq 0.9, \tag{19}$$

where $\lambda_{[i]}$ denotes the ith largest component of λ. The effect on the experiment design will be to spread out the measurements over more points (at the cost of increasing the determinant of the error covariance). (See Figures 2 and 3.)

The constraint (19) is convex; it is satisfied if and only if there exists $x \in \mathbf{R}^M$ and t such

Figure 3: Experiment design of Figure 2. The curves show the sum of the largest k components of λ as a function of k, without the 90-10 constraint ('$\times$'), and with the constraint ('$\circ$'). The constraint specifies that the sum of the largest five components should be less than 0.9, *i.e.*, the curve should avoid the area inside the dashed rectangle.

that

$$\begin{array}{l} \lfloor 0.1M \rfloor t + \sum_{i=1}^{M} x_i \leq 0.9 \\ t + x_i \geq \lambda_i, \quad i = 1, \ldots, M \\ x \geq 0 \end{array} \tag{20}$$

(see [BV95Γp.318]). One can therefore compute the D-optimal design subject to the 90-10 constraint (19) by adding the linear inequalities (20) to the constraints in (18) and solving the resulting maxdet-problem in the variables $\lambda \Gamma x \Gamma t$.

2.7 Maximum likelihood (ML) parameter estimation

ML estimation of Markov chain parameters

Consider an N-state Markov chain with transition probabilities $p_{ij} \Gamma i, j = 1, \ldots, N \Gamma$

$$p_{ij} = \mathbf{Prob}(s(k+1) = j \mid s(k) = i),$$

where $s(k) \in \{1, \ldots, N\}$ denotes the state at time k. Suppose that the probabilities p_{ij} are affine functions of some unknown parameters x: $p_{ij} = f_{ij}(x) \Gamma i, j = 1, \ldots, N$. The maximum likelihood estimate of the parameters $x \Gamma$based on an observed state sequence $a_1 \Gamma a_2 \Gamma \ldots \Gamma a_n \Gamma$

is the solution of the optimization problem

$$\begin{array}{ll} \text{maximize} & \prod_{i=1}^{n-1} f_{a_i a_{i+1}}(x) \\ \text{subject to} & f_{ij}(x) \geq 0, \quad i,j = 1,\ldots,N, \\ & \sum_{j=1}^{N} f_{ij}(x) = 1, \quad i = 1,\ldots,N, \end{array}$$

which is a maxdet-problem with diagonal matrix $G(x)$.

ML estimation of the parameters of an exponential distribution

Let $y^{(1)}\Gamma..\Gamma y^{(N)} \in \mathbf{R}^n_+$ be N vectors drawn independently from a probability distribution with density

$$p(x) = \left(\prod_{i=1}^{n} \lambda_i\right) e^{-\lambda^T x}$$

on $\mathbf{R}^n_+\Gamma$where $\lambda > 0$. The maximum likelihood estimate for λ based on these N samplesΓ *i.e.*$\Gamma$the value of λ that maximizes the log-likelihood function

$$\log \prod_{i=1}^{N} p(y^{(i)}) = -\lambda^T \sum_{i=1}^{N} y^{(i)} + N \sum_{i=1}^{n} \log \lambda_i \tag{21}$$

can be easily found by differentiation: $\lambda_j = N/\sum_{k=1}^{N} y_j^{(k)}\Gamma j = 1,\ldots,n$. Note that problem (21) is an example of an unconstrained maxdet-problem (5). We can therefore add LMI constraints that express prior informationΓas in

$$\begin{array}{ll} \text{maximize} & -\left(\frac{1}{N}\sum_{i=1}^{N} y^{(i)}\right)^T \lambda + \sum_{i=1}^{n} \log \lambda_i \\ \text{subject to} & \lambda > 0 \\ & A\lambda = b \\ & C\lambda \leq d. \end{array}$$

ML estimation of structured covariance matrices

A related example is the ML estimation of structured covariance matrices of a normal distribution. This problem has a long history; see *e.g.*ΓAnderson[And69ΓAnd70].

Let $y^{(1)}\Gamma\ldots\Gamma y^{(N)}$ be N samples from a normal distribution $N(0,\Sigma)$. The ML estimate for Σ is the positive definite matrix that maximizes the log-likelihood function $\prod_{i=1}^{N} p(y^{(i)})\Gamma$ where

$$p(x) = ((2\pi)^p \det \Sigma)^{-1/2} \exp\left(-\frac{1}{2} x^T \Sigma^{-1} x\right).$$

In other words$\Gamma\Sigma$ can be found by solving

$$\begin{array}{ll} \text{maximize} & \log\det \Sigma^{-1} - \frac{1}{N}\sum_{i=1}^{N} y^{(i)T} \Sigma^{-1} y^{(i)} \\ \text{subject to} & \Sigma > 0. \end{array} \tag{22}$$

This can be expressed as a maxdet-problem in the *inverse* $R = \Sigma^{-1}$:

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr} SR + \log\det R^{-1} \\ \text{subject to} & R > 0, \end{array} \tag{23}$$

where $S = \frac{1}{N}\sum_{i=1}^{n} y^{(i)}y^{(i)^T}$. Problem (23) has the straightforward analytical solution $R = S^{-1}$ (provided S is nonsingular).

It is often useful to impose additional structure on the covariance matrix Σ or its inverse R (Anderson[And69ΓAnd70]ΓBurgΓLuenbergerΓWenger[BLW82]ΓScharf[Sch91Γ§6.13]Γ Dembo[Dem86]). In some special cases (*e.g.*ΓΣ is circulant) analytical solutions are known; in other cases where the constraints can be expressed as LMIs in RΓthe ML estimate can be obtained from a maxdet-problem. To give a simple illustrationΓbounds on the variances Σ_{ii} can be expressed as LMIs in R

$$\Sigma_{ii} = e_i^T R^{-1} e_i \leq \alpha \Longleftrightarrow \begin{bmatrix} R & e_i \\ e_i^T & \alpha \end{bmatrix} \geq 0.$$

The formulation as a maxdet-problem is also useful when the matrix S is singular (for exampleΓbecause the number of samples is too small) andΓas a consequenceΓthe maxdet-problem (23) is unbounded below. In this case we can impose constraints (*i.e.*Γprior information) on ΣΓfor example lower and upper bounds on the diagonal elements of R.

2.8 Gaussian channel capacity

The Gaussian channel and the water-filling algorithm

The entropy of a normal distribution $N(\mu, \Sigma)$ isΓup to a constantΓequal to $\frac{1}{2}\log\det\Sigma$ (see Cover and Thomas[CT91ΓChapter 9]). It is therefore not surprising that maxdet-problems arise naturally in information theory and communications. One example is the computation of channel capacity.

Consider a simple Gaussian communication channel: $y = x + v$Γwhere yΓxΓand v are random vectors in $\mathbf{R}^n$; $x \sim N(0, X)$ is the input; y is the outputΓand $v \sim N(0, R)$ is additive noiseΓindependent of x. This model can represent n parallel channelsΓor one single channel at n different time instants or n different frequencies.

We assume the noise covariance R is known and given; the input covariance X is the variable to be determinedΓsubject to constraints (such as power limits) that we will describe below. Our goal is to maximize the *mutual information* between input and outputΓgiven by

$$\frac{1}{2}(\log\det(X + R) - \log\det R) = \frac{1}{2}\log\det(I + R^{-1/2}XR^{-1/2})$$

(see [CT91]). The *channel capacity* is defined as the maximum mutual information over all input covariances X that satisfy the constraints. (ThusΓthe channel capacity depends on R and the constraints.)

The simplest and most common constraint is a limit on the average total power in the inputΓ*i.e.*Γ

$$\mathbf{E}x^T x/n = \mathbf{Tr}X/n \leq P. \tag{24}$$

The information capacity subject to this average power constraint is the optimal value of

$$\begin{array}{ll} \text{maximize} & \frac{1}{2}\log\det(I + R^{-1/2}XR^{-1/2}) \\ \text{subject to} & \mathbf{Tr}X \leq nP \\ & X \geq 0 \end{array} \tag{25}$$

(see [CT91Γ§10]). This is a maxdet-problem in the variable $X = X^T$.

There is a straightforward solution to (25)Γknown in information theory as the *water-filling* algorithm (see [CT91Γ§10]Γ[CP89]). Let $R = V\Lambda V^T$ be the eigenvalue decomposition of R. By introducing a new variable $\tilde{X} = V^T XV$Γwe can rewrite the problem as

$$\begin{array}{ll} \text{maximize} & \frac{1}{2}\log\det(I + \Lambda^{-1/2}\tilde{X}\Lambda^{-1/2}) \\ \text{subject to} & \mathbf{Tr}\tilde{X} \leq nP \\ & \tilde{X} \geq 0. \end{array}$$

Since the off-diagonal elements of $\tilde{X}$ do not appear in the constraintsΓbut decrease the objectiveΓ$\tilde{X}$ will be diagonal at the optimum. Using Lagrange multipliers one can show that the solution is $\tilde{X}_{ii} = \max(\nu - \lambda_i, 0)$Γ$i = 1, \ldots, n$Γwhere the Lagrange multiplier ν is to be determined from $\sum \tilde{X}_{ii} = nP$. The term 'water-filling' refers to a visual description of this procedure (see [CT91Γ§10]Γ[CP89]).

Average power constraints on each channel

Problem (25) can be extended and modified in many ways. For exampleΓwe can replace the average *total* power constraint by an average power constraint on the individual channelsΓ *i.e.*Γwe can replace (24) by $\mathbf{E}x_k^2 = X_{kk} \leq P$Γ$k = 1, \ldots, n$. The capacity subject to this constraint can be determined by solving the maxdet-problem

$$\begin{array}{ll} \text{maximize} & \frac{1}{2}\log\det\left(I + R^{-1/2}XR^{-1/2}\right) \\ \text{subject to} & X \geq 0 \\ & X_{kk} \leq P, \quad k = 1, \ldots, n. \end{array}$$

The water-filling algorithm does not apply hereΓbut the capacity is readily computed by solving this maxdet-problem in X. MoreoverΓwe can easily add other constraintsΓsuch as power limits on subsets of individual channelsΓor an upper bound on the correlation coefficient between two components of x:

$$\frac{|X_{ij}|}{\sqrt{X_{ii}X_{jj}}} \leq \rho_{\max} \Longleftrightarrow \begin{bmatrix} \sqrt{\rho_{\max}}X_{ii} & X_{ij} \\ X_{ij} & \sqrt{\rho_{\max}}X_{jj} \end{bmatrix} \geq 0.$$

Gaussian channel capacity with feedback

Suppose that the n components of xΓyΓand v are consecutive values in a time series. The question whether knowledge of the past values v_k helps in increasing the capacity of the channel is of great interest in information theory [CT91Γ§10.6]). In the Gaussian channel with feedback one usesΓinstead of xΓthe vector $\tilde{x} = Bv + x$ as input to the channelΓwhere B

is a strictly lower triangular matrix. The output of the channel is $y = \tilde{x} + v = x + (B+I)v$. We will assume there is an average total power constraint: $\mathbf{E}\tilde{x}^T\tilde{x}/n \leq P$.

The mutual information between $\tilde{x}$ and y is

$$\frac{1}{2}\left(\log\det((B+I)R(B+I)^T + X) - \log\det R\right),$$

so we maximize the mutual information by solving

$$\begin{array}{ll} \text{maximize} & \frac{1}{2}\left(\log\det((B+I)R(B+I)^T + X) - \log\det R\right) \\ \text{subject to} & \mathbf{Tr}(BRB^T + X) \leq nP \\ & X \geq 0 \\ & B \text{ strictly lower triangular} \end{array}$$

over the matrix variables B and X. To cast this problem as a maxdet-problemΓwe introduce a new variable $Y = (B+I)R(B+I)^T + X$ ($i.e.$Γthe covariance of y)Γand obtain

$$\begin{array}{ll} \text{maximize} & \log\det Y \\ \text{subject to} & \mathbf{Tr}(Y - RB^T - BR - R) \leq nP \\ & Y - (B+I)R(B+I)^T \geq 0 \\ & B \text{ strictly lower triangular} \end{array} \tag{26}$$

The second constraint can be expressed as an LMI in B and YΓ

$$\begin{bmatrix} Y & B+I \\ (B+I)^T & R^{-1} \end{bmatrix} > 0,$$

so (26) is a maxdet-problem in B and Y.

Capacity of channel with cross-talk

Suppose the n channels are independentΓ$i.e.$Γall covariances are diagonalΓand that the noise covariance depends on X: $R_{ii} = r_i + a_i X_{ii}$Γwith $a_i > 0$. This has been used as a model of near-end cross-talk (see [AC92]). The capacity (with the total average power constraint) is the optimal value of

$$\begin{array}{ll} \text{maximize} & \frac{1}{2}\sum_{i=1}^{n}\log\left(1 + \frac{X_{ii}}{r_i + a_i X_{ii}}\right) \\ \text{subject to} & X_{ii} \geq 0, \quad i = 1, \ldots, n \\ & \sum_{i=1}^{n} X_{ii} \leq nP, \end{array}$$

which can be cast as a maxdet-problem

$$\begin{array}{ll} \text{maximize} & \frac{1}{2}\sum_{i=1}^{n}\log(1+t_i) \\ \text{subject to} & X_{ii} \geq 0, \; t_i \geq 0, \quad i = 1, \ldots, n, \\ & \begin{bmatrix} 1 - a_i t_i & \sqrt{r_i} \\ \sqrt{r_i} & a_i X_{ii} + r_i \end{bmatrix} \geq 0, \quad i = 1, \ldots, n, \\ & \sum_{i=1}^{n} X_{ii} \leq nP. \end{array}$$

The LMI is equivalent to $t_i \leq X_{ii}/(r_i + a_i X_{ii})$. This problem can be solved using standard methods; the advantage of a maxdet-problem formulation is that we can add other (LMI) constraints on XΓ*e.g.*Γindividual power limits. As another interesting possibilityΓwe could impose constraints that distribute the power across the channels more uniformlyΓ*e.g.*Γa 90-10 type constraint (see §2.6).

2.9 Problems involving moments

Bounds on expected values via semidefinite programming

Let t be a random real variable. The expected values $\mathbf{E}t^k$ are called the (power) *moments* of the distribution of t. The following classical result gives a characterization of a moment sequence: There exists a probability distribution on $\mathbf{R}$ such that $x_k = \mathbf{E}t^k$Γ$k = 0, \ldots, 2n$Γif and only if $x_0 = 1$ and

$$H(x_0, \ldots, x_{2n}) = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_1 & x_2 & x_3 & \cdots & x_n & x_{n+1} \\ x_2 & x_3 & x_4 & \cdots & x_{n+1} & x_{n+2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{n-1} & x_n & x_{n+1} & \cdots & x_{2n-2} & x_{2n-1} \\ x_n & x_{n+1} & x_{n+2} & \cdots & x_{2n-1} & x_{2n} \end{bmatrix} \geq 0. \tag{27}$$

It is easy to see that the condition is necessary: let $x_i = \mathbf{E}t^i$Γ$i = 0, \ldots, 2n$ be the moments of some distributionΓand let $y = [y_0\ y_1\ \cdots\ y_n]^T \in \mathbf{R}^{n+1}$. Then we have

$$y^T H(x_0, \ldots, x_{2n}) y = \sum_{i,j=0}^{n} y_i y_j \mathbf{E}t^{i+j} = \mathbf{E}\left(y_0 + y_1 t^1 + \cdots + y_n t^n\right)^2 \geq 0.$$

Sufficiency is less obvious. The proof is classical (and based on convexity arguments); see *e.g.*ΓKrein and Nudelman[KN77Γp.182] or Karlin and Studden[KS66Γp.189–199]. There are similar conditions for distributions on finite or semi-infinite intervals.

Note that condition (27) is an LMI in the variables x_kΓ*i.e.*Γthe condition that x_0Γ...Γ x_{2n} be the moments of some distribution on $\mathbf{R}$ can be expressed as an LMI in x. Using this factΓwe can cast some interesting moment problems as SDPs and maxdet-problems.

Suppose t is a random variable on $\mathbf{R}$. We do not know its distributionΓbut we do know some bounds on the momentsΓ*i.e.*Γ

$$\underline{\mu}_k \leq \mathbf{E}t^k \leq \overline{\mu}_k$$

(which includesΓas a special caseΓknowing exact values of some of the moments). Let $p(t) = c_0 + c_1 t + \cdots + c_{2n} t^{2n}$ be a given polynomial in t. The expected value of $p(t)$ is linear in the moments $\mathbf{E}t^i$:

$$\mathbf{E}p(t) = \sum_{i=0}^{2n} c_i \mathbf{E}t^i = \sum_{i=0}^{2n} c_i x_i.$$

We can compute upper and lower bounds for $\mathbf{E}p(t)$Γ

$$\begin{array}{ll} \text{minimize (maximize)} & \mathbf{E}p(t) \\ \text{subject to} & \underline{\mu}_k \leq \mathbf{E}t^k \leq \overline{\mu}_k, \quad k = 1, \ldots, 2n, \end{array}$$

over all probability distributions that satisfy the given moment boundsΓby solving the SDPs

$$\begin{array}{ll} \text{minimize (maximize)} & c_1x_1 + \cdots + c_{2n}x_{2n} \\ \text{subject to} & \underline{\mu}_k \le x_k \le \overline{\mu}_k, \quad k = 1, \ldots, 2n \\ & H(1, x_1, \ldots, x_{2n}) \ge 0 \end{array}$$

over the variables $x_1\Gamma\ldots\Gamma x_{2n}$. This gives bounds on $\mathbf{E}p(t)\Gamma$over all probability distributions that satisfy the known moment constraints. The bounds are sharp in the sense that there are distributionsΓwhose moments satisfy the given moments boundsΓfor which $\mathbf{E}p(t)$ takes on the upper and lower bounds found by these SDPs.

A related problem was considered by DahlquistΓ Eisenstatt and Golub[DEG72]Γ who analytically compute bounds on $\mathbf{E}t^{-1}$ and $\mathbf{E}t^{-2}\Gamma$given the moments $\mathbf{E}t^i\Gamma i = 1, \ldots, n$. (Here t is a random variable in a finite interval.) Using semidefinite programming one can solve more general problems where upper and lower bounds on $\mathbf{E}t^i\Gamma i = 1\ldots, n\Gamma$(or the expected value of some polynomials) are known.

Another application arises in the steady-state analysis of continuous-time Markov processesΓin which some (multivariable) moments can be computed cheaply (Schwerer[Sch96]).

Upper bound on the variance via semidefinite programming

As another exampleΓone can maximize the variance of $t\Gamma$over all probability distributions that satisfy the moment constraints (to obtain a sharp upper bound on the variance of t):

$$\begin{array}{ll} \text{maximize} & \mathbf{E}t^2 - (\mathbf{E}t)^2 \\ \text{subject to} & \underline{\mu}_k \le \mathbf{E}t^k \le \overline{\mu}_k, \quad k = 1, \ldots, 2n, \end{array}$$

which is equivalent to the SDP

$$\begin{array}{ll} \text{maximize} & y \\ \text{subject to} & \begin{bmatrix} x_2 - y & x_1 \\ x_1 & 1 \end{bmatrix} \ge 0 \\ & \underline{\mu}_k \le x_k \le \overline{\mu}_k, \quad k = 1, \ldots, 2n \\ & H(1, x_1, \ldots, x_{2n}) \ge 0 \end{array}$$

with variables $y\Gamma x_1\Gamma\ldots\Gamma x_{2n}$. The 2×2-LMI is equivalent to $y \le x_2 - x_1^2$. More generallyΓ one can compute an upper bound on the variance of a given polynomial $\mathbf{E}p(t)^2 - (\mathbf{E}p(t))^2$. Thus we can compute an upper bound on the variance of a polynomial $p(t)\Gamma$given some bounds on the moments.

A robust estimate of the moments

Another interesting problem is the maxdet-problem

$$\begin{array}{ll} \text{maximize} & \log\det H(1, x_1, \ldots, x_{2n}) \\ \text{subject to} & \underline{\mu}_k \le x_k \le \overline{\mu}_k, k = 1, \ldots, 2n \\ & H(1, x_1, \ldots, x_{2n}) > 0. \end{array} \tag{28}$$

The solution can serve as a 'robust' solution to the feasibility problem of finding a probability distribution that satisfies given bounds on the moments. While the SDPs provide lower and upper bounds on $\mathbf{E}p(t)$Γthe maxdet-problem should provide a reasonable guess of $\mathbf{E}p(t)$.

Note that the maxdet-problem (28) is equivalent to

$$\begin{array}{ll} \text{maximize} & \log\det \mathbf{E}f(t)f(t)^T \\ \text{subject to} & \underline{\mu} \leq \mathbf{E}f(t) \leq \overline{\mu} \end{array} \tag{29}$$

over all probability distributions on $\mathbf{R}$Γwhere $f(t) = [1\ t\ t^2\ \ldots\ t^n]^T$. We can interpret this as the problem of designing a random experiment to estimate the coefficients of a polynomial $p(t) = c_0 + c_1 t + \cdots + c_n t^n$. (We ignore numerical issues such as ill-conditioning.)

2.10 Quasi-Newton updates

In quasi-Newton methods for unconstrained minimization of a convex function fΓthe Newton step $-\nabla^2 f(x)^{-1}\nabla f(x)$ is replaced by $-H^{-1}\nabla f(x)$Γwhere $H = H^T > 0$ is an approximation of the Hessian matrixΓbased on prior information and previous gradient evaluations. In each iterationΓas the algorithm moves from x to the next point x^+Γa new approximation H^+ is determinedΓbased on the current HΓand on the difference between the gradients at x^+ and x. A good updating rule for H should satisfy several properties: H^+ should be close to HΓit should be easy to compute (orΓmore preciselyΓthe search direction $-H^{+^{-1}}\nabla f(x^+)$ should be easy to compute)Γand it should incorporate the new information obtained by evaluating the gradient $\nabla f(x^+)$. This last property is usually enforced by imposing the secant condition

$$H^+(x^+ - x) = \nabla f(x^+) - \nabla f(x). \tag{30}$$

Byrd and Nocedal[BN89] have proposed to measure the difference between H and H^+ by using the Kullback-Leibler divergence (or relative entropy)Γgiven by

$$\frac{1}{2}\left(\mathbf{Tr}\, H^{-1/2}H^+H^{-1/2} - \log\det H^{-1/2}H^+H^{-1/2} - n\right)$$

(see also Dennis and Wolkowicz[DW93] and Lewis[Lew96]). The Kullback-Leibler divergence is nonnegative for all positive definite H and H^+Γand zero only if $H^+ = H$. Computing the update that satisfies the secant condition and minimizes the Kullback-Leibler divergence is a maxdet-problem in H^+:

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, H^{-1/2}H^+H^{-1/2} - \log\det H^{-1/2}H^+H^{-1/2} - n \\ \text{subject to} & H^+ > 0 \\ & H^+(x^+ - x) = \nabla f(x^+) - \nabla f(x). \end{array} \tag{31}$$

Fletcher[Fle91] has shown that the solution is given by

$$H^+ = H - \frac{Hss^TH}{s^THs} + \frac{gg^T}{s^Tg}, \tag{32}$$

assuming that $s^Tg > 0$Γwhere $s = x^+ - x$ and $g = \nabla f(x^+) - \nabla f(x)$. Formula (32) is well known in unconstrained optimization as the BFGS (BroydenΓFletcherΓGoldfarbΓShanno) quasi-Newton update.

Fletcher's observation opens the possibility of adding more complicated LMI constraints to the maxdet-problem (31)Γand solving the resulting problem numerically. For exampleΓ one can impose a certain sparsity pattern on H^+Γor one can relax the secant condition as

$$\|H^+(x^+ - x) - \nabla f(x^+) + \nabla f(x)\| \leq \epsilon,$$

where ϵ is a given tolerance.

Updating H^+ by numerically solving a maxdet-problem will obviously involve far more computation than the BFGS update. ThusΓa general maxdet-problem formulation for quasi-Newton updates is only interesting when gradient evaluations are very expensive.

3 The dual problem

We associate with (1) the *dual* problem

$$\begin{array}{ll} \text{maximize} & \log\det W - \mathbf{Tr} G_0 W - \mathbf{Tr} F_0 Z + l \\ \text{subject to} & \mathbf{Tr} G_i W + \mathbf{Tr} F_i Z = c_i, \quad i = 1, ..., m, \\ & W = W^T > 0, \quad Z = Z^T \geq 0. \end{array} \tag{33}$$

The variables are $W \in \mathbf{R}^{l\times l}$ and $Z \in \mathbf{R}^{n\times n}$. Problem (33) is also a maxdet-problemΓand can be converted into a problem of the form (1) by elimination of the equality constraints.

We say W and Z are dual feasible if they satisfy the constraints in (33)Γand strictly dual feasible if in addition $Z > 0$. We also refer to the maxdet-problem (1) as the primal problem and say x is primal feasible if $F(x) \geq 0$ and $G(x) > 0$Γand strictly primal feasible if $F(x) > 0$ and $G(x) > 0$.

Let p^* and d^* be the optimal values of problem (1) and (33)Γrespectively (with the convention that $p^* = +\infty$ if the primal problem is infeasibleΓand $d^* = -\infty$ if the dual problem is infeasible).

Theorem 1 *$p^* \geq d^*$. If (1) is strictly feasible, the dual optimum is achieved; if (33) is strictly feasible, the primal optimum is achieved. In both cases, $p^* = d^*$.*

The theorem follows from standard results in convex analysis (Rockafellar[Roc70])Γso we will not prove it here. We willΓhoweverΓshow the first partΓ*i.e.*Γthat $p^* \geq d^*$ always holds. Suppose x is primal feasibleΓand WΓZ are dual feasible. We will show that the primal objective evaluated at x is greater than the dual objective evaluated at WΓZ:

$$c^T x + \log\det G(x)^{-1} \geq \log\det W - \mathbf{Tr} G_0 W - \mathbf{Tr} F_0 Z + l,$$

and as a consequenceΓ$p^* \geq d^*$. We have

$$\begin{aligned} & c^T x + \log\det G(x)^{-1} + \log\det W^{-1} + \mathbf{Tr} G_0 W + \mathbf{Tr} F_0 Z - l \\ = & \sum_{i=1}^m x_i \mathbf{Tr} G_i W + \mathbf{Tr} G_0 W + \sum_{i=1}^m x_i \mathbf{Tr} F_i Z + \mathbf{Tr} F_0 Z - \log\det G(x) W - l \\ = & \mathbf{Tr} G(x) W - \log\det G(x) W - l + \mathbf{Tr} F(x) Z. \end{aligned} \tag{34}$$

This expression is always nonnegative: the last term$\Gamma\,\mathbf{Tr}F(x)Z\Gamma$is nonnegative because $\mathbf{Tr}AB \geq 0$ when $A = A^T \geq 0$ and $B = B^T \geq 0$; the sum of the first three terms can be written as

$$\mathbf{Tr}G(x)W - \log\det G(x)W - l = \mathbf{Tr}W^{1/2}G(x)W^{1/2} - \log\det W^{1/2}G(x)W^{1/2} - l,$$

which is twice the Kullback-Leibler divergence between W and $G(x)$. It is nonnegative since $\log\det A^{-1} \geq -\mathbf{Tr}A + l$ for positive definite $A = A^T \in \mathbf{R}^{l\times l}$ (as can be verified by taking the eigenvalue decomposition of A and using the inequality $\log x \leq x - 1$ for $x > 0$).

The difference between the primal and dual objectiveΓ*i.e.*Γexpression (34)$\Gamma$is called the *duality gap* associated with $x\Gamma W$ and Z. Theorem 1 states that the duality gap is always nonnegative$\Gamma$and zero only if $x\Gamma W$ and Z are optimal.

Note that zero duality gap (34) implies $G(x)W = I$ and $F(x)Z = 0$. This gives the optimality condition for the maxdet-problem (1): a primal feasible x is optimal if there exists a $Z \geq 0\Gamma$such that $F(x)Z = 0$ and

$$\mathbf{Tr}G_iG(x)^{-1} + \mathbf{Tr}F_iZ = c_i, \quad i = 1, \ldots, m.$$

This optimality condition is always sufficient; it is also necessary if the primal problem is strictly feasible.

In the remainder of the paper we will assume that the maxdet-problem is strictly primal and dual feasible. By Theorem 1Γthis assumption implies that the primal problem is bounded below and the dual problem is bounded above$\Gamma$with equality at the optimum$\Gamma$and that the primal and dual optimal sets are nonempty.

Example: semidefinite programming dual

As an illustrationΓwe derive from (33) the dual problem for the SDP (2). Substituting $G_0 = 1\Gamma G_i = 0\Gamma n = 1\Gamma$in (33) yields

$$\begin{array}{ll} \text{maximize} & \log W - W - \mathbf{Tr}F_0Z + 1 \\ \text{subject to} & \mathbf{Tr}F_iZ = c_i, \quad i = 1, \ldots, m, \\ & W > 0, \quad Z \geq 0. \end{array}$$

The optimal value of W is oneΓso the dual problem reduces to

$$\begin{array}{ll} \text{maximize} & -\mathbf{Tr}F_0Z \\ \text{subject to} & \mathbf{Tr}F_iZ = c_i, \quad i = 1, \ldots, m, \\ & Z \geq 0, \end{array}$$

which is the dual SDP (in the notation used in [VB96]).

Example: D-optimal experiment design

As a second example we derive the dual of the experiment design problem (18). After a few simplifications we obtain

$$\begin{array}{ll} \text{maximize} & \log\det W + p - z \\ \text{subject to} & W = W^T > 0 \\ & v^{(i)T}Wv^{(i)} \leq z, \; i = 1, \ldots, M, \end{array} \tag{35}$$

where the variables are the matrix W and the scalar variable z. Problem (35) can be further simplified. The constraints are homogeneous in W and $z\Gamma$so for each dual feasible $W\Gamma z$ we have a ray of dual feasible solutions $tW\Gamma tz\Gamma t > 0$. It turns out that we can analytically optimize over t: replacing W by tW and z by tz changes the objective to $\log\det W + p\log t + p - tz\Gamma$which is maximized for $t = p/z$. After this simplification$\Gamma$and with a new variable $\tilde{W} = (p/z)W\Gamma$problem (35) becomes

$$\begin{array}{ll} \text{maximize} & \log\det\tilde{W} \\ \text{subject to} & \tilde{W} > 0 \\ & v^{(i)T}\tilde{W}v^{(i)} \leq p, \; i = 1, \ldots, M. \end{array} \tag{36}$$

Problem (36) has an interesting geometrical meaning: the constraints state that $\tilde{W}$ determines an ellipsoid $\{x \mid x^T\tilde{W}x \leq p\}\Gamma$centered at the origin$\Gamma$that contains the points $v^{(i)}\Gamma$ $i = 1, \ldots, M$; the objective is to maximize $\det\tilde{W}\Gamma i.e.\Gamma$to minimize the volume of the ellipsoid.

There is an interesting connection between the optimal primal variables λ_i and the points $v^{(i)}$ that lie on the boundary of the optimal ellipsoid $\mathcal{E}$. First note that the duality gap associated with a primal feasible λ and a dual feasible $\tilde{W}$ is equal to

$$\log\det\left(\sum_{i=1}^{M}\lambda_i v^{(i)}v^{(i)T}\right)^{-1} - \log\det\tilde{W},$$

and is zero (hence$\Gamma\lambda$ is optimal) if and only if $\tilde{W} = \left(\sum_{i=1}^{M}\lambda_i v^{(i)}v^{(i)T}\right)^{-1}$. Hence$\Gamma\lambda$ is optimal if

$$\mathcal{E} = \left\{ x \in \mathbf{R}^p \;\middle|\; x^T\left(\sum_{i=1}^{M}\lambda_i v^{(i)}v^{(i)T}\right)^{-1} x \leq p \right\}$$

is the minimum-volume ellipsoidΓcentered at the originΓthat contains the points $v^{(j)}\Gamma j = 1, \ldots, M$. We also have (in fact$\Gamma$for any feasible λ)

$$\sum_{j=1}^{M}\lambda_j\left(p - v^{(j)T}\left(\sum_{i=1}^{M}\lambda_i v^{(i)}v^{(i)T}\right)^{-1} v^{(j)}\right) = p - \mathbf{Tr}\left(\sum_{j=1}^{M}\lambda_j v^{(j)}v^{(j)T}\right)\left(\sum_{i=1}^{M}\lambda_i v^{(i)}v^{(i)T}\right)^{-1} = 0. \tag{37}$$

If λ is optimalΓthen each term in the sum on the left hand side is positive (since $\mathcal{E}$ contains all vectors $v^{(j)}$)$\Gamma$and therefore the sum can only be zero if each term is zero:

$$\lambda_j > 0 \Longrightarrow v^{(j)T}\left(\sum_{i=1}^{M}\lambda_i v^{(i)}v^{(i)T}\right)^{-1} v^{(j)} = p,$$

Geometrically$\Gamma\lambda_j$ is nonzero only if $v^{(j)}$ lies on the boundary of the minimum volume ellipsoid. This makes more precise the intuitive idea that an optimal experiment only uses 'extreme' test vectors. Figure 4 shows the optimal ellipsoid for the experiment design example of Figure 2.

Figure 4: In the dual of the D-optimal experiment design problem we compute the minimum-volume ellipsoid, centered at the origin, that contains the test vectors. The test vectors with a nonzero weight lie on the boundary of the optimal ellipsoid. Same data and notation as in Figure 2.

The duality between D-optimal experiment designs and minimum-volume ellipsoids also extends to non-finite compacts sets (Titterington[Tit75]Γ Pronzato and Walter[PW94]). The D-optimal experiment design problem on a compact set $C \subset \mathbf{R}^p$ is

$$\text{maximize} \ \log\det \mathbf{E} vv^T \tag{38}$$

over all probability measures on C. This is a convex but semi-infinite optimization problemΓ with dual ([Tit75])

$$\begin{array}{ll} \text{maximize} & \log\det \tilde{W} \\ \text{subject to} & \tilde{W} > 0 \\ & v^T \tilde{W} v \leq p, \quad v \in C. \end{array} \tag{39}$$

AgainΓ we see that the dual is the problem of computing the minimum volume ellipsoidΓ centered at the originΓ and covering the set C.

General methods for solving the semi-infinite optimization problems (38) and (39) fall outside the scope of this paper. In particular casesΓ howeverΓ these problems can be solved as maxdet-problems. One interesting example arises when C is the union of a finite number of ellipsoids. In this caseΓ the dual (39) can be cast as a maxdet-problem (see §2.1) and hence efficiently solved; by dualityΓ we can recover from the dual solution the probability distribution that solves (38).

4 The central path

In this section we describe the *central path* of the maxdet-problem (1)Γ and give some of its properties. The central path plays a key role in interior point methods for the maxdet-problem.

Definition

For strictly feasible x and $t \geq 1$ we define

$$\varphi_p(t,x) \triangleq t\left(c^T x + \log\det G(x)^{-1}\right) + \log\det F(x)^{-1}. \tag{40}$$

This function is the sum of two convex functions: the first term is a positive multiple of the objective function in (1); the second term $\log\det F(x)^{-1}$ is a barrier function for the set $\{x \mid F(x) > 0\}$. For future use we note that the gradient and Hessian of $\varphi_p(x,t)$ are given by the expressions

$$\begin{aligned}(\nabla\varphi_p(t,x))_i &= t\left(c_i - \mathbf{Tr}G(x)^{-1}G_i\right) - \mathbf{Tr}F(x)^{-1}F_i, &(41)\\ \left(\nabla^2\varphi_p(t,x)\right)_{ij} &= t\mathbf{Tr}G(x)^{-1}G_iG(x)^{-1}G_j + \mathbf{Tr}F(x)^{-1}F_iF(x)^{-1}F_j, &(42)\end{aligned}$$

for $i,j = 1,\ldots,m$.

It can be shown that $\varphi_p(t,x)$ is a *strictly* convex function of x if the m matrices $\mathbf{diag}(G_i,F_i)$, $i = 1,\ldots,m$, are linearly independent, and that it is bounded below (since we assume the problem is strictly dual feasible; see the appendix). We define $x^\star(t)$ as the unique minimizer of $\varphi_p(t,x)$:

$$x^\star(t) = \operatorname{argmin}\left\{\varphi_p(t,x) \mid G(x) > 0,\ \ F(x) > 0\right\}.$$

The curve $x^\star(t)$, parametrized by $t \geq 1$, is called the *central path*.

The dual central path

Points $x^\star(t)$ on the central path are characterized by the optimality conditions $\nabla\varphi_p(t,x^\star(t)) = 0$, *i.e.*, using the expression (41)

$$\mathbf{Tr}G(x^\star(t))^{-1}G_i + \frac{1}{t}\mathbf{Tr}F(x^\star(t))^{-1}F_i = c_i,\ \ i = 1,\ldots,m.$$

From this we see that the matrices

$$W^\star(t) = G(x^\star(t))^{-1}, \quad Z^\star(t) = \frac{1}{t}F(x^\star(t))^{-1} \tag{43}$$

are strictly dual feasible. The duality gap associated with $x^\star(t)$, $W^\star(t)$ and $Z^\star(t)$ is, from expression (34),

$$\mathbf{Tr}F(x^\star(t))Z^\star(t) + \mathbf{Tr}G(x^\star(t))W^\star(t) - \log\det G(x^\star(t))W^\star(t) - l = \frac{n}{t},$$

which shows that $x^\star(t)$ converges to the solution of the maxdet-problem as $t \to \infty$.

It can be shown that the pair $(W^\star(t), Z^\star(t))$ actually lies on the *dual* central path, defined as

$$(W^\star(t), Z^\star(t)) = \operatorname{argmin}\left\{\varphi_d(t,W,Z) \;\middle|\; \begin{array}{l} W = W^T \geq 0,\ \ Z = Z^T > 0,\\ \mathbf{Tr}G_iW + \mathbf{Tr}F_iZ = c_i,\ \ i = 1,\ldots,m\end{array}\right\}$$

where

$$\varphi_d(t,W,Z) \triangleq t\left(\log\det W^{-1} + \mathbf{Tr}G_0W + \mathbf{Tr}F_0Z - l\right) + \log\det Z^{-1}.$$

The close connections between primal and dual central path are summarized in the following theorem.

Theorem 2 *Let x be strictly primal feasible, and W, Z strictly dual feasible. Then*

$$\varphi_p(t,x) + \varphi_d(t,W,Z) \geq n(1+\log t) \tag{44}$$

with equality if and only if $x = x^\star(t)$, $W = W^\star(t)$, $Z = Z^\star(t)$.

Proof. If $A = A^T \in \mathbf{R}^{p\times p}$ and $A > 0\Gamma$ then $-\log\det A \geq -\mathbf{Tr}A + p$ (by convexity of $-\log\det A$ on the cone of positive semidefinite matrices). Applying this inequalityΓwe find

$$\begin{aligned}
\varphi_p(t,x) + \varphi_d(t,W,Z) &= t\left(\mathbf{Tr}G(x)W + \mathbf{Tr}F(x)Z - \log\det G(x)W - l\right) - \log\det F(x)Z \\
&= t\left(-\log\det W^{1/2}G(x)W^{1/2} + \mathbf{Tr}W^{1/2}G(x)W^{1/2}\right) \\
&\quad - \log\det tZ^{1/2}F(x)Z^{1/2} + \mathbf{Tr}tZ^{1/2}F(x)Z^{1/2} + n\log t - tl \\
&\geq tl + n + n\log t - tl = n(1+\log t).
\end{aligned}$$

The equality for $x = x^\star(t)\Gamma W = W^\star(t)\Gamma Z = Z^\star(t)$ can be verified by substitution.

Tangent to the central path

We conclude this section by describing how the tangent direction to the central path can be computed. Let $\phi_1(x) = -\log\det G(x)$ and $\phi_2(x) = -\log\det F(x)$. A point $x^\star(t)$ on the central path is characterized by

$$t\left(c + \nabla\phi_1(x^\star(t))\right) + \nabla\phi_2(x^\star(t)) = 0.$$

The tangent direction $\frac{\partial x^\star(t)}{\partial t}$ can be found by differentiating with respect to t:

$$\left(c + \nabla\phi_1(x^\star(t))\right) + \left(t\nabla^2\phi_1(x^\star(t)) + \nabla^2\phi_2(x^\star(t))\right)\frac{\partial x^\star(t)}{\partial t} = 0,$$

so that

$$\frac{\partial x^\star(t)}{\partial t} = -\left(t\nabla^2\phi_1(x^\star(t)) + \nabla^2\phi_2(x^\star(t))\right)^{-1}\left(c + \nabla\phi_1(x^\star(t))\right). \tag{45}$$

The tangent direction can also be expressed as the solution of a least-squares problem; see Appendix A.

By differentiating (43)Γwe obtain the tangent to the dual central pathΓ

$$\frac{\partial W^\star(t)}{\partial t} = -G(x^\star(t))^{-1}\left(\sum_{i=1}^m \frac{\partial x^\star{}_i(t)}{\partial t}G_i\right)G(x^\star(t))^{-1}, \tag{46}$$

$$\frac{\partial Z^\star(t)}{\partial t} = -\frac{1}{t^2}F(x^\star(t))^{-1} - \frac{1}{t}F(x^\star(t))^{-1}\left(\sum_{i=1}^m \frac{\partial x^\star{}_i(t)}{\partial t}F_i\right)F(x^\star(t))^{-1}. \tag{47}$$

5 Newton's method

In this section we consider the problem of minimizing $\varphi_p(t,x)$ for fixed $t\Gamma$ *i.e.*Γcomputing $x^\star(t)\Gamma$given a strictly feasible initial point:

$$\begin{array}{ll}
\text{minimize} & \varphi_p(t,x) \\
\text{subject to} & G(x) > 0 \\
& F(x) > 0.
\end{array} \tag{48}$$

This includesΓas a special caseΓthe analytic centering problem ($t = 1$ and $F(x) = 1$). Our main motivation for studying (48) will become clear in next sectionΓwhen we discuss an interior-point method based on minimizing $\varphi_p(t, x)$ for a sequence of values t.

Newton's method with line search can be used to solve problem (48) efficiently.

Newton method for minimizing $\varphi_p(t, x)$
given strictly feasible xΓtolerance δ ($0 < \delta \leq 0.5$)
repeat

1. Compute the Newton direction $\delta x^N = -\left(\nabla^2 \varphi_p(t, x)\right)^{-1} \nabla \varphi_p(t, x)$
2. Compute $\lambda = (\delta x^N \nabla^2 \varphi_p(t, x) \delta x^N)^{1/2}$
3. **if** ($\lambda > 0.5$)Γcompute $\widehat{h} = \operatorname{argmin} \varphi_p(t, x + h\delta x^N)$
   else $\widehat{h} = 1$
4. Update: $x := x + \widehat{h}\delta x^N$

until $\lambda \leq \delta$

The quantity

$$\lambda = (\delta x^N \nabla^2 \varphi_p(t, x) \delta x^N)^{1/2} \tag{49}$$

is called the *Newton decrement* at x. The cost of Step 3 (the *line search*) is very smallΓ usually negligible compared with the cost of computing the Newton direction; see §8 for details.

It is well known that the asymptotic convergence of Newton's method is quadratic. Nesterov and Nemirovsky in [NN94Γ§2.2] give a complete analysis of the *global* speed of convergence. The main result is the following theorem.

Theorem 3 *The algorithm terminates in less than*

$$11(\varphi_p(t, x^{(0)}) - \varphi_p(t, x^\star(t))) + \log_2 \log_2(1/\delta) \tag{50}$$

iterations, and when it terminates, $\varphi_p(t, x) - \varphi_p(t, x^\star(t)) \leq \delta$.

A self-contained proof is given in the appendix.

Note that the right-hand side of (50) does not depend on the problem size (*i.e.*ΓmΓnΓor l) at allΓand only depends on the problem data through the difference between the value of the function $\varphi_p(t, \cdot)$ at the initial point $x^{(0)}$ and at the central point $x^\star(t)$.

The term $\log_2 \log_2(1/\delta)$Γwhich is characteristic of quadratic convergenceΓgrows *extremely slowly* with required accuracy δ. For all practical purposes it can be considered a constantΓ sayΓ five (which guarantees an accuracy of $\delta = 2.33 \cdot 10^{-10}$). Not quite preciselyΓ thenΓ the theorem says we can compute $x^\star(t)$ in at most $11(\varphi_p(t, x^{(0)}) - \varphi_p(t, x^\star(t))) + 5$ Newton steps. The precise statement is that within this number of iterations we can compute an extremely good approximation of $x^\star(t)$. In the sequelΓ we will speak of 'computing the central point $x^\star(t)$' when we really mean computing an extremely good approximation. We can justify this on several grounds. It is possible to adapt our exposition to account for the extremely small approximation error incurred by terminating the Newton process after $11(\varphi_p(t, x^{(0)}) - \varphi_p(t, x^\star(t))) + 5$ steps. IndeedΓthe errors involved are certainly on the same

Figure 5: Number of Newton iterations to minimize $\log\det A(x)^{-1}$ versus $\log\det A(x^{(0)})^{-1} - \log\det A(x^\star)^{-1}$ (with $\delta = 2.33 \cdot 10^{-10}$, *i.e.*, $\log_2\log_2(1/\delta) = 5$). Random matrix completion problems of three sizes ('+': $m = 20$; $l = 20$, '$\times$': $m = 20$, $l = 100$, '$\circ$': $m = 50$, $l = 100$). The dotted line is a least-squares fit of the data and is given by $3 + 0.67(\log\det A(x^{(0)})^{-1} - \log\det A(x^\star)^{-1})$. The dashed line is the upper bound of Theorem 3 ($5 + 11(\log\det A(x^{(0)})^{-1} - \log\det A(x^\star)^{-1})$).

scale as computer arithmetic (roundoff) errorsΓso if a complexity analysis is to be carried out with such precisionΓit should also account for roundoff error.

We will see during the course of the proof (in the appendix) that Theorem 3 holds for an 'implementable' version of the algorithm as wellΓin which an appropriate *approximate* line search is used instead of the exact line search.

Numerical experiment

The bound provided by Theorem 3 on the number of Newton steps required to compute $x^\star(t)\Gamma$starting from $x^{(0)}\Gamma$will play an important role in our path-following method. It is therefore useful to examine how the bound compares to the actual number of Newton steps required in practice to compute $x^\star(t)$.

Figure 5 shows the results of a numerical experiment that compares the actual convergence of Newton's method with the bound (50). The test problem is a matrix completion problem

$$\begin{array}{ll} \text{minimize} & \log\det A(x)^{-1} \\ \text{subject to} & A(x) = A_f + \sum_{k=1}^{m} x_k \left(E_{i_k j_k} + E_{j_k i_k}\right) > 0, \end{array}$$

which is a particular case of (48) with $c = 0\Gamma\, G(x) = A(x)\Gamma\, F(x) = 1\Gamma$and $\varphi_p(t,x) = \log\det A(x)^{-1}$. We considered problems of three different sizes: $m = 20\Gamma l = 20$ (indicated by '+'); $m = 20\Gamma l = 100$ (indicated by '$\times$'); $m = 50\Gamma l = 100$ (indicated by '$\circ$'). Each point

on the figure corresponds to a different problem instanceΓgenerated as follows.

- The matrices A_f were constructed as $A_f = UU^T$ with the elements of U drawn from a normal distribution $N(0,1)$. This guarantees that $x = 0$ is strictly feasible. The m index pairs $(i_k, j_k)\Gamma i_k \neq j_k\Gamma$were chosen randomly with a uniform distribution over the off-diagonal index pairs. For each of the three problem sizes$\Gamma 50$ instances were generated.

- For each problem instanceΓwe first computed $x^\star$ using $x = 0$ as starting point. We then selected a value γ (uniformly in the interval $(0, 30)$)Γgenerated a random $\widehat{x} \in \mathbf{R}^m$ (with distribution $N(0, I)$)$\Gamma$and then computed $x^{(0)} = x^\star + t(\widehat{x} - x^\star)$ such that

$$\log\det(A(x^{(0)}))^{-1} - \log\det(A(x^*))^{-1} = \gamma.$$

  This point $x^{(0)}$ was used as starting point for the Newton algorithm.

Our experience with other problems shows that the results for this family of random problems are quite typical.

From the results we can draw two important conclusions.

- The quantity $\log\det(A(x^{(0)}))^{-1} - \log\det(A(x^\star))^{-1}$ not only provides an upper bound on the number of Newton iterations via Theorem 3; it is also a very good predictor of the number of iterations in practice. The dimensions m and l on the other hand have much less influence (except of courseΓthrough $\log\det(A(x^{(0)}))^{-1} - \log\det(A(x^\star))^{-1}$).

- The average number of Newton iterations seems to grow as

$$\alpha + \beta\left(\log\det(A(x^{(0)}))^{-1} - \log\det(A(x^\star))^{-1}\right),$$

  with $\alpha \simeq 3\Gamma\beta \simeq 0.7$. This is significantly smaller than the upper bound of Theorem 3 ($\alpha = 5\Gamma\beta = 11$).

In summaryΓwe conclude that the difference $\varphi_p(t, x^{(0)}) - \varphi_p(t, x^\star(t))$ is a good measureΓ in theory and in practice$\Gamma$of the effort required to compute $x^\star(t)$ using Newton's methodΓ starting at $x^{(0)}$.

A computable upper bound on the number of Newton steps

Note that $\varphi_p(t, x^\star(t))$ is not known explicitly as a function of t. To evaluate the bound (50) one has to compute $x^\star(t)\Gamma i.e.\Gamma$carry out the Newton algorithm. (WhichΓat the very leastΓ would seem to defeat the purpose of trying to estimate or bound the number of Newton steps required to compute $x^\star(t)$.) Therefore the bound of Theorem 3 is not (directly) useful in practice. From Theorem 2ΓhoweverΓit follows that every dual feasible point $W\Gamma Z$ provides a lower bound for $\varphi_p(t, x^\star(t))$:

$$\varphi_p(t, x^\star(t)) \geq -\varphi_d(t, W, Z) + n(1 + \log t).$$

and that the bound is exact if $W = W^\star(t)$ and $Z = Z^\star(t)$.

We can therefore replace the bound (50) by a weaker, but more easily computed bound, provided we have a dual feasible pair W, Z:

$$11(\varphi_p(t, x^{(0)}) - \varphi_p(t, x^\star(t))) + \log_2 \log_2(1/\delta) \leq 11\psi_{\mathrm{ub}}(t, x^{(0)}, W, Z) + \log_2 \log_2(1/\delta), \quad (51)$$

where

$$\psi_{\mathrm{ub}}(t, x, W, Z) = \varphi_p(t, x) + \varphi_d(t, W, Z) - n(1 + \log t). \quad (52)$$

This is the bound we will use in practice (and in our complexity analysis): it gives a readily computed bound on the number of Newton steps required to compute $x^\star(t)$, starting from $x^{(0)}$, given any dual feasible W, Z.

6 Path-following algorithms

Path-following methods for convex optimization have a long history. In their 1968 book [FM68], Fiacco and MacCormick work out many general properties, *e.g.*, convergence to an optimal point, connections with duality, etc. No attempt was made to give a worst-case convergence analysis, until Renegar[Ren88] proved polynomial convergence of a path-following algorithm for linear programming. Nesterov and Nemirovsky[NN94, §3] studied the convergence for nonlinear convex problems and provided proofs of polynomial worst-case complexity. See [NN94, pp.379–386] and Den Hertog[dH93] for a historical overview.

We will present two variants of a path-following method for the maxdet-problem. The short-step version of §6.2 is basically the path-following method of [FM68, NN94], with a simplified, self-contained complexity analysis. In the long-step verion of §6.3 we combine the method with predictor steps to accelerate convergence. This, too, is a well known technique, originally proposed by Fiacco and MacCormick; our only addition is a new step selection rule.

6.1 General idea

One iteration proceeds as follows. The algorithm starts at a point $x^\star(t)$ on the central path. As we have seen above, the duality gap associated with $x^\star(t)$ is n/t. We then select a new value $t^+ > t$, and choose a strictly feasible starting point $\widehat{x}$ (which may or may not be equal to $x^\star(t)$). The point $\widehat{x}$ serves as an approximation of $x^\star(t^+)$ and is called the *predictor* of $x^\star(t^+)$. Starting at the predictor $\widehat{x}$, the algorithm computes $x^\star(t^+)$ using Newton's method. This reduces the duality gap by a factor t^+/t. The step from $x^\star(t)$ to $x^\star(t^+)$ is called an *outer iteration.*

The choice of t^+ and $\widehat{x}$ involves a tradeoff. A large value of t^+/t means fast duality gap reduction, and hence fewer outer iterations. On the other hand it makes it more difficult to find a good predictor $\widehat{x}$, and hence more Newton iterations may be needed to compute $x^\star(t^+)$.

In the method discussed below, we impose a bound on the maximum number of Newton iterations per outer iteration, by requiring that the predictor $\widehat{x}$ and the new value of t^+ satisfy

$$\varphi_p(t^+, \widehat{x}) - \varphi_p(t^+, x^\star(t^+)) \leq \gamma. \quad (53)$$

This implies that no more than $5 + 11\gamma$ Newton iterations are required to compute $x^\star(t^+)$ starting at $\hat{x}$. Of course, the exact value of the left hand side is not known, unless we carry out the Newton minimization, but as we have seen above, we can replace the condition by

$$\psi_{\mathrm{ub}}(t^+, \hat{x}, \widehat{W}, \widehat{Z}) = \gamma, \tag{54}$$

where $\widehat{W}$ and $\widehat{Z}$ are conveniently chosen dual feasible points.

The parameters in the algorithm are $\gamma > 0$ and the desired accuracy ϵ.

Path-following algorithm

given $\gamma > 0, t \geq 1, x := x^\star(t)$
repeat
1. Select $t^+, \hat{x}, \widehat{W}, \widehat{Z}$ such that $t^+ > t$ and $\psi_{\mathrm{ub}}(t^+, \hat{x}, \widehat{W}, \widehat{Z}) = \gamma$
2. Compute $x^\star(t^+)$ starting at $\hat{x}$, using the Newton algorithm of §5
3. $t := t^+, x := x^\star(t^+)$

until $n/t \leq \epsilon$

Step 1 in this outline is not completely specified. In the next sections we will discuss in detail different choices. We will show that one can always find $\hat{x}, \widehat{W}, \widehat{Z}$ and t^+ that satisfy

$$\frac{t^+}{t} \geq 1 + \sqrt{\frac{2\gamma}{n}}. \tag{55}$$

This fact allows us to estimate the total complexity of the method, *i.e.*, to derive a bound on the total number of Newton iterations required to reduce the duality gap to ϵ. The algorithm starts on the central path, at $x^\star(t^{(0)})$, with initial duality gap $\epsilon^{(0)} = n/t^{(0)}$. Each iteration reduces the duality gap by t^+/t. Therefore the total number of outer iterations required to reduce the initial gap of $\epsilon^{(0)}$ to a final value below ϵ is at most

$$\left\lceil \frac{\log(\epsilon^{(0)}/\epsilon)}{\log(1 + \sqrt{2\gamma/n})} \right\rceil \leq \left\lceil \sqrt{n} \frac{\log(\epsilon^{(0)}/\epsilon)}{\log(1 + \sqrt{2\gamma})} \right\rceil.$$

(The inequality follows from the concavity of $\log(1+x)$.) The total number of Newton steps can therefore be bounded as

$$\text{Total \#Newton iterations} \leq \lceil 5 + 11\gamma \rceil \left\lceil \sqrt{n} \frac{\log(\epsilon^{(0)}/\epsilon)}{\log(1 + \sqrt{2\gamma})} \right\rceil = O\left(\sqrt{n} \log(\epsilon^{(0)}/\epsilon)\right). \tag{56}$$

This upper bound increases slowly with the problem dimensions: it grows as $\sqrt{n}$, and is independent of l and m. We will see later that the performance in practice is even better.

Note that we assume that the minimization in Step 2 of the algorithm is exact. The justification of this assumption lies in the very fast local convergence of Newton's method: we have seen in §5 that it takes only a few iterations to improve a solution with Newton decrement $\lambda \leq 0.5$ to one with a very high accuracy.

Nevertheless, in a practical implementation (as well as in a rigorous theoretical analysis), one has to take into account the fact that $x^\star(t)$ can only be computed approximately. For

example, the stopping criterion $n/t \leq \epsilon$ is based on the duality gap associated with exactly central points $x^\star(t)$, $W^\star(t)$, and $Z^\star(t)$, and is therefore not quite accurate if $x^\star(t)$ is only known approximately. We give a suitably modified criterion in Appendix A, where we show that dual feasible points are easily computed during the centering step (Step 2) once the Newton decrement is less than one. Using the associated duality gap yields a completely rigorous stopping criterion. We will briefly point out some other modifications, as we develop different variants of the algorithm in the next sections; full details are described in Appendix A. With these modifications, the algorithm works well even when $x^\star(t)$ is computed approximately. (We often use a value $\delta = 10^{-3}$ in the Newton algorithm.)

It is also possible to extend the simple worst-case complexity analysis to take into account incomplete centering, but we will not attempt such an analysis here.

6.2 Fixed-reduction algorithm

The simplest variant uses $\widehat{x} = x^\star(t)$, $\widehat{W} = W^\star(t)$, and $\widehat{Z} = Z^\star(t)$ in Step 1 of the algorithm. Substitution in condition (54) gives

$$\begin{aligned}
&\psi_{\mathrm{ub}}(t^+, \widehat{x}, \widehat{W}, \widehat{Z}) \\
&\quad = t^+(\mathbf{Tr}\, G(x^\star(t))W^\star(t) + \mathbf{Tr}\, F(x^\star(t))Z^\star(t) - \log\det G(x^\star(t))W^\star(t) - l) \\
&\qquad - \log\det F(x^\star(t))Z^\star(t) - n(1 + \log t^+) \\
&\quad = n(t^+/t - 1 - \log(t^+/t)) = \gamma,
\end{aligned} \tag{57}$$

which is a simple nonlinear equation in one variable, with a unique solution $t^+ > t$. We call this variant of the algorithm the *fixed-reduction* algorithm because it uses the same value of t^+/t — and hence achieves a fixed duality gap reduction factor — in each outer iteration. The outline of the fixed-reduction algorithm is as follows.

Fixed-reduction algorithm

given $\gamma > 0$, $t \geq 1$, $x := x^\star(t)$
Find α such that $n(\alpha - 1 - \log\alpha) = \gamma$
repeat
1. $t^+ := \alpha t$
2. Compute $x^\star(t^+)$ starting at x, using the Newton algorithm of §5
3. $t := t^+$, $x := x^\star(t^+)$

until $n/t \leq \epsilon$

We can be brief in the convergence analysis of the method. Each outer iteration reduces the duality gap by a factor α, so the number of outer iterations is *exactly*

$$\left\lceil \frac{\log(\epsilon^{(0)}/\epsilon)}{\log\alpha} \right\rceil.$$

The inequality (55), which was used in the complexity analysis of previous section, follows from the fact that for $y \geq 1$

$$n(y - 1 - \log y) \leq \frac{n}{2}(y-1)^2,$$

and hence $\alpha \geq 1 + \sqrt{2\gamma/n}$.

This convergence analysis also reveals the limitation of the fixed reduction method: the number of outer iterations is never better than the number predicted by the theoretical analysis. The upper bound on the total number of Newton iterations (56) is also a good estimate in practice provided we replace the constant $5+11\gamma$ with an empirically determined estimate such as $3+0.7\gamma$ (see Figure 5). The purpose of next section is to develop a method with the same worst-case complexity as the fixed-reduction algorithm but a much better performance in practice.

6.3 Primal-dual long-step algorithm

It is possible to use much larger values of t^+/t and hence achieve larger gap reduction per outer iteration by using a better choice for $\widehat{x}$, $\widehat{W}$ and $\widehat{Z}$ in Step 1 of the path-following algorithm.

A natural choice for $\widehat{x}$ is to take a point along the tangent to the central path, *i.e.*,

$$\widehat{x} = x^\star(t) + p\frac{\partial x^\star(t)}{\partial t},$$

for some $p > 0$, where the tangent direction is given by (45). Substitution in (54) gives a nonlinear equation from which t^+ and p can be determined. Taking the idea one step further, one can allow $\widehat{W}$ and $\widehat{Z}$ to vary along the tangent to the dual central path, *i.e.*, take

$$\widehat{W} = W^\star(t) + q\frac{\partial W^\star(t)}{\partial t}, \quad \widehat{Z} = Z^\star(t) + q\frac{\partial Z^\star(t)}{\partial t}$$

for some $q > 0$, with the tangent directions given by (46) and (47). Equation (54) then has three unknowns: t^+, the primal step length p, and the dual step length q. The fixed-reduction update of previous section uses the solution $t^+ = \alpha t$, $p = q = 0$; an efficient method for finding a solution with larger t^+ is described below.

The outline of the long-step algorithm is as follows.

Primal-dual long-step algorithm

given $\gamma > 0$, $t \geq 1$, $x := x^\star(t)$, $W := W^\star(t)$, $Z := Z^\star(t)$
Find α such that $n(\alpha - 1 - \log\alpha) = \gamma$
repeat

1. *Compute tangent to central path.* $\delta x := \frac{\partial x^\star(t)}{\partial t}$, $\delta W := \frac{\partial W^\star(t)}{\partial t}$, $\delta Z := \frac{\partial Z^\star(t)}{\partial t}$
2. *Parameter selection and predictor step.*
   2a. $t^+ := \alpha t$
   repeat {
      2b. $\widehat{p}, \widehat{q} = \mathrm{argmin}_{p,q}\ \psi_{ub}(t^+, x + p\delta x, W + q\delta W, Z + q\delta Z)$
      2c. Compute t^+ from $\psi_{\mathrm{ub}}(t^+, x + \widehat{p}\delta x, W + \widehat{q}\delta W, Z + \widehat{q}\delta Z) = \gamma$
   }
   2d. $\widehat{x} = x + \widehat{p}\delta x$
3. *Centering step.* Compute $x^\star(t^+)$ starting at $\widehat{x}$, using the Newton algorithm of §5
4. *Update.* $t := t^+$, $x := x^\star(t^+)$, $W := W^\star(t^+)$, $Z := Z^\star(t^+)$

until $n/t \leq \epsilon$

Figure 6: Parameter selection and predictor step in long-step algorithm alternates between minimizing $\psi_{\rm ub}(t^+, p, q)$ over primal step length p and dual step length q, and then increasing t^+ until $\psi_{\rm ub}(t^+, p, q) = \gamma$.

Again we assume exact centering in Step 3. In practice, approximate minimization works, provided one includes a small correction to the formulas of the tangent directions; see Appendix A.

Step 2 computes a solution to (54), using a technique illustrated in Figure 6. The figure shows four iterations of the inner loop of Step 2 (for an instance of the problem family described in §9). With a slight abuse of notation, we write $\psi_{\rm ub}(t^+, p, q)$ instead of

$$\psi_{\rm ub}(t^+, x^\star(t) + p\delta x, W^\star(t) + q\delta W, Z^\star(t) + q\delta Z). \tag{58}$$

We start at the value $t^{(0)} = t$, at the left end of the horizontal axis. The first curve (marked $\psi_{\rm ub}(t^+, 0, 0)$) shows (58) as a function of t^+, with $p = q = 0$, which simplifies to

$$\psi_{\rm ub}(t^+, x^\star(t), W^\star(t), Z^\star(t)) = n(t^+/t - 1 - \log(t^+/t))$$

(see §6.2). This function is equal to zero for $t^+ = t$, and equal to γ for the short-step update $t^+ = \alpha t$. We then do the first iteration of the inner loop of Step 2. Keeping t^+ fixed at its value $t^{(1)}$, we minimize the function (58) over p and q (Step 2b). This produces new values $\widehat{p} = p^{(1)}$ and $\widehat{q} = q^{(1)}$ with a value of $\psi_{\rm ub} < \gamma$. This allows us to increase t^+ again (Step 2c). The second curve in the figure (labeled $\psi_{\rm ub}(t^+, p^{(1)}, q^{(1)})$) shows the function (58) as a function of t^+ with fixed values $p = p^{(1)}, q = q^{(1)}$. The intersection with $\psi_{\rm ub} = \gamma$ gives the next value $t^+ = t^{(2)}$.

These two steps (2b, 2c) are repeated either for a fixed number of iterations or until t^+ converges (which in the example of Figure 6 happens after four or five iterations). Note

that in each step 2c, we increase t^+, so that in particular, the final value of t^+ will be at least as large as its initial (short-step) value, $t^+ = \alpha t$. Thus, the complexity analysis for the short-step method still applies.

In practice, the inner loop (2b, 2c) often yields a value of t^+ considerably larger than the short-step value αt, while maintaining the same upper bound on the number of Newton step required to compute the next iterate $x^\star(t^+)$. In the example shown in the figure, the final value of t^+ is about a factor of 2.5 larger than the short-step value; in general, a factor of 10 is not uncommon.

Using some preprocessing we will describe in §8, the cost of the inner loop (2b, 2c) is very small, in most cases negligible compared with the cost of computing the tangent vectors.

Finally, we note that the dual variables Z and W in the primal-dual long-step algorithm are only used to allow larger step sizes.

7 Preliminary phases

The algorithm starts at a central point $x^\star(t)$, for some $t \geq 1$. In this section we discuss how to select the initial t, and how to compute such a point.

Feasibility

If no strictly primal feasible point is known, one has to precede the algorithm with a first phase to solve the (SDP) feasibility problem: Find x that satisfies $G(x) > 0$, $F(x) > 0$. More details can be found in [VB96].

Choice of initial t

We now consider the situation where a strictly primal feasible point $x^{(0)}$ is known, but $x^{(0)}$ is not on the central path. In that case one has to select an appropriate initial value of t and compute a central point by Newton's method starting at $x^{(0)}$. In theory (and often in practice) the simple choice $t = 1$ works.

It is not hard, however, to imagine cases where the choice $t = 1$ would be inefficient in practice. Suppose, for example, that the initial $x^{(0)}$ is very near $x^\star(100)$, so a reasonable initial value of t is 100 (but we don't know this). If we set $t = 1$, we expend many Newton iterations 'going backwards' up the central path towards the point $x^\star(1)$. Several outer iterations, and many Newton steps later, we find ourselves back near where we started, around $x^\star(100)$.

If strictly dual feasible points $W^{(0)}, Z^{(0)}$ are known, then we start with a known duality gap α associated with $x^{(0)}, W^{(0)}$ and $Z^{(0)}$. A very reasonable initial choice for t is then $t = \max\{1, n/\alpha\}$, since when $t = n/\alpha$, the centering stage computes central points with the same duality gap as the initial primal and dual solutions. In particular, the preliminary centering stage does not increase the duality gap (as it would in the scenario sketched above).

We can also interpret and motivate the initial value $t = n/\alpha$ in terms of the function $\psi_{\text{ub}}(t, x^{(0)}, W^{(0)}, Z^{(0)})$, which provides an upper bound on the number of Newton steps re-

quired to compute $x^\star(t)$ starting at $x^{(0)}$. From the definition (52) we have

$$\psi_{\mathrm{ub}}(t, x^{(0)}, W^{(0)}, Z^{(0)}) = t\alpha + \log\det F(x^{(0)})^{-1} + \log\det Z^{(0)^{-1}} - n(1 + \log t),$$

which shows that the value $t = n/\alpha$ minimizes $\psi_{\mathrm{ub}}(t, x^{(0)}, W^{(0)}, Z^{(0)})$. Thus$\Gamma$ the value $t = n/\alpha$ is the value which minimizes the upper bound on the number of Newton steps required in the preliminary centering stage.

A heuristic preliminary stage

When no initial dual feasible $Z\Gamma W$ (and hence duality gap) are knownΓ choosing an appropriate initial value of t can be difficult. We have had practical success with a variation on Newton's method that adapts the value of t at each step based on the (square of) the Newton decrement $\lambda(x,t)\Gamma$

$$\lambda(x,t)^2 = \nabla\varphi_p(t,x)^T \left(\nabla^2\varphi_p(t,x)\right)^{-1} \nabla\varphi_p(t,x),$$

which serves as a measure of proximity to the central path. It is a convex function of $t\Gamma$and is readily minimized in t for fixed x.

Our heuristic preliminary phase is:

Preliminary centering phase

given strictly feasible x
$t := 1$
repeat {
  1. $t := \max\{1, \mathrm{argmin}\lambda(x,t)\}$
  2. $\delta x = -\left(\nabla^2\varphi_p(t,x)\right)^{-1} \nabla\varphi_p(t,x)$
  3. $\widehat{h} = \mathrm{argmin}\ \varphi_p(t, x + h\delta x^N)$
} **until** $\lambda \le \delta$

ThusΓwe adjust t each iteration to make the Newton decrement for the current x as small as possible (subject to the condition that we not decrease t).

8 Efficient line and plane searches

In this section we describe some simple preprocessing that allows us to implement the line search in the Newton method of §5 and the plane search of §6.3 very efficiently.

Line search in Newton's method

We first consider the line search in Newton's method of §5. Let $\lambda_k\Gamma k = 1, \ldots, l\Gamma$be the generalized eigenvalues of the pair $\sum_{i=1}^m \delta x_i^N G_i\Gamma G(x)\Gamma$and $\lambda_k\Gamma k = l+1, \ldots, l+n\Gamma$be the

generalized eigenvalues of the pair $\sum_{i=1}^m \delta x_i^N F_i \Gamma F(x) \Gamma$where δx^N is the Newton direction at x. We can write $\varphi_p(t, x + h\delta x^N)$ in terms of these eigenvalues as

$$f(h) = \varphi_p(t, x + h\delta x^N) = \varphi_p(t, x) + hc^T \delta x^N + t \sum_{k=1}^{l} \log \frac{1}{1 + h\lambda_k} + \sum_{k=l+1}^{l+n} \log \frac{1}{1 + h\lambda_k}.$$

Evaluating the first and second derivatives $f'(h) \Gamma f''(h)$ of this (convex) function of $h \in \mathbf{R}$ requires only $O(n + l)$ operations (once the generalized eigenvalues λ_i have been computed). In most cases$\Gamma$the cost of the preprocessingΓ*i.e.*Γcomputing the generalized eigenvalues $\lambda_i \Gamma$ exceeds the cost of minimizing over $h \Gamma$but is small compared with the cost of computing the Newton direction. The function $\varphi_p(t, x + h\delta x^N)$ can therefore be efficiently minimized using standard line search techniques.

Plane search in long-step path-following method

A similar idea applies to the plane search of §6.3. In Step 2c of the primal-dual long-step algorithm we minimize the function $\psi_{\text{ub}}(t, x + p\delta x, W + q\delta W, Z + q\delta Z)$ over p and $q \Gamma$where $\delta x \Gamma \delta W \Gamma \delta Z$ are tangent directions to the central path. We can again reduce the function to a convenient form

$$\begin{aligned} &\psi_{\text{ub}}(t, x + p\delta x, W + q\delta W, Z + q\delta Z) \\ &= \psi_{\text{ub}}(t, x, W, Z) + p\beta_1 + q\beta_2 + t \sum_{k=1}^{l} \log \frac{1}{1 + p\lambda_k} + \sum_{k=l+1}^{l+n} \log \frac{1}{1 + p\lambda_k} \\ &\quad + t \sum_{k=1}^{l} \log \frac{1}{1 + q\mu_k} + \sum_{k=l+1}^{l+n} \log \frac{1}{1 + q\mu_k}, \end{aligned} \tag{59}$$

where $\lambda_k \Gamma k = 1, \ldots, l \Gamma$are the generalized eigenvalues of the pair $\sum_{i=1}^m \delta x_i G_i \Gamma G(x)$ and $\lambda_k \Gamma k = l + 1, \ldots, l + n \Gamma$are the generalized eigenvalues of the pair $\sum_{i=1}^m \delta x_i F_i \Gamma F(x)$; $\mu_k \Gamma$ $k = 1, \ldots, l \Gamma$are the generalized eigenvalues of the pair $\delta W \Gamma W \Gamma$and $\mu_k \Gamma k = l + 1, \ldots, l + n \Gamma$ are the generalized eigenvalues of the pair $\delta Z \Gamma Z$. The coefficients β_1 and β_2 are

$$\beta_1 = c^T \delta x, \quad \beta_2 = \mathbf{Tr} G_0 \delta W + \mathbf{Tr} F_0 \delta Z.$$

The first and second derivatives of the function (59) with respect to p and q can again be computed at a low cost of $O(l + n) \Gamma$and therefore the minimum of ψ_{ub} over the plane can be determined very cheaplyΓonce the generalized eigenvalues have been computed.

In summary$\Gamma$the cost of line or plane search is basically the cost of preprocessing (computing certain generalized eigenvalues)Γwhich is usually negligible compared to the rest of algorithm (*e.g.*Γdetermining a Newton or tangent direction).

One implication of efficient line and plane searches is that the total number of Newton steps serves as a good measure of the overall computing effort.

Figure 7: Duality gap versus number of Newton steps for randomly generated maxdet-problems of dimension $l = 10$, $n = 10$, $m = 10$. *Left:* $\gamma = 10$. *Right:* $\gamma = 50$. The crosses are the results for the fixed-reduction method; the circles are the results for the long-step method. Every cross/circle represents the gap at the end of an outer iteration.

9 Numerical examples

Typical convergence

The first experiment (Figure 7) compares the convergence of the fixed-reduction method and the long-step method. The lefthand plot shows the convergence of both methods for $\gamma = 10$; the righthand plot shows the convergence for $\gamma = 50$. Duality gap is shown vertically on a logarithmic scale ranging from 10^0 at the top to 10^{-9} at the bottom; the horizontal axis is the total number of Newton steps. Each outer iteration is shown as a symbol on the plot ('$\circ$' for the long-step and '$\times$' for the short-step method). ThusΓthe horizontal distance between two consecutive symbols shows directly the number of Newton steps required for that particular outer iteration; the vertical distance shows directly the duality gap reduction factor t^+/t.

Problem instances were generated as follows: $G_0 \in \mathbf{R}^{l\times l}\Gamma F_0 \in \mathbf{R}^{n\times n}$ were chosen random positive definite (constructed as U^TU with the elements of U drawn from a normal distribution $N(0,1)$); the matrices $G_i\Gamma F_i\Gamma i = 1,\ldots,m\Gamma$were random symmetric matricesΓwith elements drawn from $N(0,1)$; $c_i = \mathbf{Tr}G_i + \mathbf{Tr}F_i\Gamma i = 1,\ldots,m$. This procedure ensures that the problem is primal and dual feasible ($x = 0$ is primal feasible; $Z = I\Gamma W = I$ is dual feasible)Γand hence bounded below. We start on the central pathΓwith initial duality gap one.

We can make the following observations.

- The convergence is very similar over all problem instances. The number of iterations required to reduce the duality gap by a factor 1000 ranges between 5 and 50. As expectedΓthe long-step method performs much better than the fixed-reduction methodΓ and typically converges in less than 15 iterations.

- The fixed-reduction method converges almost linearly. The duality gap reduction t^+/t

per outer iteration can be computed from equation (57): $t^+/t = 3.14$ for $\gamma = 10$Γ and $t^+/t = 8.0$ for $\gamma = 50$. The number of Newton iterations per outer iteration is less than five in almost all casesΓwhich is much less than the upper bound $5 + 11\gamma$. (Remember that this bound is a combination of two conservative estimates: Theorem 3 is conservative; see Figure 5. In addition we have replaced (53) with the weaker condition (54).)

- The long-step method takes a few more Newton iterations per centering stepΓbut achieves a much larger duality gap reduction. Moreover the convergence accelerates near the optimum.

- Increasing γ has a large effect on the fixed-reduction methodΓbut only little effect on the long-step method.

Complexity versus problem size

Figure 8 shows the influence of the problem dimension on the convergence. For each triplet $(m\Gamma n\Gamma l)$ we generated 10 problem instances as above. We plot the number of Newton iterations to reduce the duality gap by a factor 1000Γstarting with duality gap 1. The plot shows the average number of Newton steps and the standard deviation. The top curve shows the results for the fixed-reduction methodΓthe lower curve is for the long-step method.

- The number of Newton iterations in the short-step method depends on n as $O(\sqrt{n})$. This is easily explained from the convergence analysis of §6.2: We have seen that the number of outer iterations grows as $\sqrt{n}$Γin theory and in practiceΓand hence the practical behavior of the fixed-reduction method is very close to the worst case behavior.

- We see that the number of iterations for the long-step method lies between 5 and 20Γ and is very weakly dependent on problem size.

Figure 9 shows similar results for a family of experiment design problems (18) in $\mathbf{R}^{10}$Γ including an 90-10 constraint (19). The points $v^{(i)}\Gamma i = 1, \ldots, M$Γwere generated from a normal distribution $N(0, I)$ on $\mathbf{R}^p$. Note that the dimensions of the corresponding maxdet-problem are $m = 2M\Gamma n = 3M + 1\Gamma l = p$. Figure 9 confirms the conclusions of the previous experiment: It shows that the complexity of the fixed-reduction method grows as $\sqrt{n}$Γwhile the complexity of the long-step method is almost independent of problem size.

10 Conclusion

The maxdet-problem (1) is a (quite specific) convex extension of the semidefinite programming problemΓand hence includes a wide variety of convex optimization problems as special cases. Perhaps more importantlyΓmaxdet-problems arise naturally in many areasΓincluding computational geometryΓlinear algebraΓexperiment designΓlinear estimationΓand information and communication theory. We have described several of these applications.

Figure 8: Newton iterations versus problem size for family of random problems. Fixed reduction method (top curve) and long-step method (lower curve). $\gamma = 10$. *Left.* $l = 10$, $n = 5$–50, $m = 10$. *Middle.* $l = 5$–50, $n = 10$, $m = 10$. *Right.* $l = 10$, $n = 10$, $m = 5$–50. The curves give the average over 10 problem instances. The error bars indicate the standard deviation.

Figure 9: Newton iterations versus problem size for family of experiment design problems of §2.6 including 90-10 rule. Fixed reduction method (top curve) and long-step method (lower curve). $\gamma = 10$, $p = 10$, $M = 15$–50. The curves show the average over 10 problem instances. The error bars indicate the standard deviation.

Some of the applications have been studied extensively in the literature, and in some cases analytic solutions or efficient specialized algorithms have been developed. We have presented an interior-point algorithm that solves *general* maxdet-problems efficiently. The method can be applied to solve maxdet-problems for which no specialized algorithm is known; in cases where such a method exists, it opens the possibility of adding useful LMI constraints, which is an important advantage in practice.

We have proved a worst-case complexity of $O(\sqrt{n})$ Newton iterations. Numerical experiments indicate that the behavior is much better in practice: the method typically requires a number of iterations that lies between 5 and 50, almost independently of problem dimension. The total computational effort is therefore determined by the amount of work per iteration, *i.e.*, the computation of the Newton directions, and therefore depends heavily on the problem structure. When no structure is exploited, the Newton directions can be computed from the least-squares formulas in Appendix A, which requires $O((n^2 + l^2)m^2)$ operations, but important savings are possible whenever we specialize the general method of this paper to a specific problem class.

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A Formulas

This appendix contains some additional computational details. We give alternative expressions for the Newton direction$\Gamma$for the dual update$\Gamma$and for the tangent directions to the primal and dual central paths. These expressions are valid even when the centering step is incomplete.

Newton direction

We first derive a least-squares formula for the Newton direction. Let $C_1 = C_1^T \in \mathbf{R}^{l\times l}$ and $C_2 = C_2^T \in \mathbf{R}^{n\times n}$ be two matrices that satisfy

$$t\mathbf{Tr}C_1G_i + \mathbf{Tr}C_2F_i = tc_i, \quad i = 1, \ldots, m.$$

Such matrices C_1 and C_2 exist since we assume that the matrices $\mathbf{diag}(G_i, F_i)\Gamma i = 1, \ldots, m\Gamma$ are linearly independent. A possible choice is $C_1 = W$ and $C_2 = tZ\Gamma$where W and Z are a dual feasible pair. Note however that C_1 and C_2 do not have to be positive definite.

The Newton direction v^N can be computed as the solution of the least-squares problem

$$\text{minimize } t\left\|I - \tilde{C}_1 - \sum_j v_j\tilde{G}_i\right\|_F^2 + \left\|I - \tilde{C}_2 - \sum_j v_j\tilde{F}_i\right\|_F^2 \tag{60}$$

where

$$\tilde{C}_1 = G(x)^{1/2}C_1G(x)^{1/2}, \quad \tilde{C}_2 = F(x)^{1/2}C_2F(x)^{1/2},$$

and for $j = 1, \ldots, m$

$$\tilde{G}_j = G(x)^{-1/2}G_jG(x)^{-1/2}, \quad \tilde{F}_j = F(x)^{-1/2}F_jF(x)^{-1/2}.$$

The norm $\|A\|_F$ in (60) is the Frobenius-norm$\Gamma\|A\|_F = \left(\mathbf{Tr}A^TA\right)^{1/2}$.

One can verify that the solution of (60) is the Newton direction by writing out the optimality conditions (normal equations). For $i = 1, \ldots, m\Gamma$we have

$$\begin{aligned} 0 &= t\mathbf{Tr}\tilde{G}_i \left(I - \tilde{C}_1 - \sum_{j=1}^m v_j^N \tilde{G}_j \right) + \mathbf{Tr}\tilde{F}_i \left(I - \tilde{C}_2 - \sum_{j=1}^m v_j^N \tilde{F}_j \right) \\ &= t\mathbf{Tr}G_i G(x)^{-1} - t\mathbf{Tr}G_i C_1 - t\sum_{j=1}^m v_j^N \mathbf{Tr}G_i G(x)^{-1} G_j G(x)^{-1} \\ &\quad + \mathbf{Tr}F_i F(x)^{-1} - \mathbf{Tr}F_i C_2 - \sum_{j=1}^m v_j^N \mathbf{Tr}F_i F(x)^{-1} F_j F(x)^{-1} \end{aligned}$$

or

$$-t\left(c_i - \mathbf{Tr}G_i G(x)^{-1}\right) - \mathbf{Tr}F_i F(x)^{-1} = \sum_{j=1}^m v_j^N \left(t\mathbf{Tr}G_i G(x)^{-1} G_j G(x)^{-1} + \mathbf{Tr}F_i F(x)^{-1} F_j F(x)^{-1} \right). \tag{61}$$

This last equation is the definition of the Newton direction $-\nabla\varphi_p(x,t) = \nabla^2\varphi_p(x,t)v^N$.

The *Newton decrement* at x is defined as

$$\begin{aligned} \lambda &= \left((v^N)^T \nabla^2 \phi_p(x,t) v^N \right)^{1/2} \\ &= \left(t \left\| \sum_{i=1}^m v_i^N G(x)^{-1/2} G_i G(x)^{-1/2} \right\|_F^2 + \left\| \sum_{i=1}^m v_i^N F(x)^{-1/2} F_i F(x)^{-1/2} \right\|_F^2 \right)^{1/2} \\ &= \left(t \sum_{i=1}^n \lambda_i^2 + \sum_{i=1}^l \gamma_i^2 \right)^{1/2} \end{aligned} \tag{62}$$

where λ_i are the generalized eigenvalues of the pair $\sum_j v_j^N G_j \Gamma G(x)$ and γ_i are the generalized eigenvalues of the pair $\sum_j v_j^N F_i \Gamma F(x)$.

Dual update

The least-squares problem (60) can also be written as a set of linear equations with as variables $v^N\Gamma$and two symmetric matrices $U^N \in \mathbf{R}^{l\times l}\Gamma V^N \in \mathbf{R}^{n\times n}$:

$$\frac{1}{t} G(x) U^N G(x) + \sum_{i=1}^m v_i^N G_i = -G(x) C_1 G(x) + G(x) \tag{63}$$

$$F(x) V^N F(x) + \sum_{i=1}^m v_i^N F_i = -F(x) C_2 F(x) + F(x) \tag{64}$$

$$\mathbf{Tr}G_i U^N + \mathbf{Tr}F_i V^N = 0, \quad i = 1, \ldots, m. \tag{65}$$

The equivalence of this system of $m + l(l+1)/2 + n(n+1)/2$ equations with the least-squares problem (60) follows from the optimality conditions of (60). One obtains the normal equations (61) by eliminating W^N and Z^N from (63) and (64) and substituting in (65).

Equations (63)–(64) can be rewritten as

$$\frac{1}{t}U^N + C_1 = G(x)^{-1} - \sum_{i=1}^{m} v_i^N G(x)^{-1} G_i G(x)^{-1} \tag{66}$$

$$V^N + C_2 = F(x)^{-1} - \sum_{i=1}^{m} v_i^N F(x)^{-1} F_i F(x)^{-1}. \tag{67}$$

From this we can make the following observations. Equation (65) and the definition of C_1 and C_2 imply that the matrices $W \triangleq \frac{1}{t}U^N + C_1 \Gamma Z \triangleq \frac{1}{t}(V^N + C_2)$ satisfy the equalities

$$\mathbf{Tr} G_i W + \mathbf{Tr} F_i Z = c_i, \quad i = 1, \ldots, m.$$

MoreoverΓW and Z are also positive definite if the Newton decrement is less than one: by (62)$\Gamma \lambda < 1$ implies that all generalized eigenvalues of the matrix pairs $\sum_j v_j^N G_j \Gamma G(x)$ and $\sum_j v_j^N F_j \Gamma F(x)$ are less than one$\Gamma$and therefore the matrices on the right-hand side of equations (66) and (67) are positive definite.

The important conclusion is that we do not need the exact value of $x^\star(t)$ to obtain dual feasible points. As soon as the Newton decrement is less than oneΓdual feasible solutions $W \Gamma Z$ result as a byproduct of computing the Newton direction. On the central path the Newton decrement is zero and the dual variables reduce to $W^\star(t) = G(x^\star(t))^{-1} \Gamma Z^\star(t) = \frac{1}{t} F(x^\star(t) x)^{-1}$. In practice it is impossible to compute $x^\star(t)$ exactly$\Gamma$and one should therefore use the formulas (66) and (67) instead of (43).

Tangent to primal central path

There is a similar least-squares formula for tangent direction to the central path. Let $x = x^\star(t) \Gamma W = W^\star(t) \Gamma$and $Z = Z^\star(t)$. The vector $v^p = \frac{\partial x^\star(t)}{\partial t}$ is the solution of the least-squares problem

$$\text{minimize } t \left\| I - \tilde{W} - \sum_{j=1}^{m} v_j \tilde{G}_j \right\|_F^2 + \left\| -t\tilde{Z} - \sum_{j=1}^{m} v_j \tilde{F}_j \right\|_F^2. \tag{68}$$

where $\tilde{W} = G(x)^{1/2} W G(x)^{1/2}$ and $\tilde{Z} = F(x)^{1/2} Z F(x)^{1/2}$. This can be verified by working out the normal equations: for $i = 1, \ldots, m \Gamma$we have

$$\begin{aligned} 0 &= t\mathbf{Tr}\tilde{G}_i \left(I - \tilde{W} - \sum_{j=1}^{m} v_j^p \tilde{G}_j \right) + \mathbf{Tr}\tilde{F}_i \left(-t\tilde{Z} - \sum_{j=1}^{m} v_j^p \tilde{F}_j \right) \\ &= t \left(\mathbf{Tr} G_i G(x)^{-1} - \mathbf{Tr} G_i W - \sum_{j=1}^{m} v_j^p \mathbf{Tr} G(x)^{-1} G_i G(x)^{-1} G_j \right) \\ &\quad - t\mathbf{Tr} F_i Z - \sum_{j=1}^{m} v_j^p \mathbf{Tr} F(x)^{-1} F_i F(x)^{-1} F_j \end{aligned}$$

or

$$-t(c_i - \mathbf{Tr} G_i G(x)^{-1}) = \sum_{j=1}^{m} v_j^p \left(t\mathbf{Tr} G_i G(x)^{-1} G_j G(x)^{-1} + \mathbf{Tr} F_i F(x)^{-1} F_j F(x)^{-1} \right)$$

which isΓup to the factor $t \Gamma$equal to (45).

Tangent to dual central path

The tangent to the dual central path can be derived from the solution v^p in a very similar way as the dual update above. We write the least-squares problem as a set of linear equations in the variables $v^p \Gamma U^p$ and $V^p \Gamma$

$$\begin{aligned} \frac{1}{t} G(x) U^p G(x) + \sum_{i=1}^{m} v_i^p G_i &= -G(x) W G(x) + G(x) \\ F(x) V^p F(x) + \sum_{i=1}^{m} v_i^p F_i &= -t F(x) Z F(x) \\ \mathbf{Tr} G_i U^p + \mathbf{Tr} F_i V^p &= 0, \quad i = 1, \ldots, m. \end{aligned}$$

We can write $U^p \Gamma V^p$ as

$$U^p = t \left(-W + G(x)^{-1} - \sum_j v_j^p G(x)^{-1} G_j G(x)^{-1} \right) \tag{69}$$

$$V^p = -tZ - \sum_j v_j^p F(x)^{-1} F_j F(x)^{-1}, \tag{70}$$

and using the expressions $W = G(x)^{-1}$ and $Z = (1/t) F(x)^{-1} \Gamma$we see that U^p and V^p are are identical to $\frac{\partial W^\star(t)}{\partial t}$ and $\frac{\partial Z^\star(t)}{\partial t}$ (see (46) and (47))Γagain up to a factor t.

Note in deriving the expressions for primal and dual tangent directions we have assumed that the current $x \Gamma W \Gamma$and Z are exactly centered. If $x \Gamma W \Gamma$and Z are not exactly on the central pathΓwe can still compute a primal search direction from (68)$\Gamma$and a dual search direction from (69)–(70).

B Complexity of Newton's method

In this appendix we analyze the complexity of Newton's method for minimizing

$$\phi_p(t, x) = t(c^T x + \log \det G(x)^{-1}) + \log \det F(x)^{-1}$$

subject to $F(x) > 0$ and $G(x) > 0 \Gamma$and for fixed $t \geq 1$. We follow Nesterov and Todd[NT94]Γ and Nesterov and Nemirovsky[NN94].

Region of quadratic convergence

Theorem 4 *Let λ be the Newton decrement at x. If $\lambda < 1$, then $x^+ = x - \nabla^2 \phi_p(t, x)^{-1} \nabla \phi_p(t, x)$ is strictly feasible, and $\lambda^+ \leq \lambda^2$, where λ^+ is the Newton decrement at x^+.*

In other words we have quadratic convergence in the region $\lambda < 1$.

Proof. Let $v^N = -\nabla^2 \phi_p(t, x)^{-1} \nabla \phi_p(t, x)$ be the Newton step. As beforeΓ we write $\lambda_i \Gamma i = 1, \ldots, l \Gamma$for the generalized eigenvalues of the matrix pair $\sum_j v_j^N G_j \Gamma G(x) \Gamma$and $\gamma_i \Gamma$ $i = 1, \ldots, n \Gamma$for the generalized eigenvalues of the matrix pair $\sum_j v_j^N F_j \Gamma F(x)$.

It follows from (62) that the eigenvalues λ_i and γ_i are less than one if $\lambda < 1$. Therefore we also have that

$$G(x^+) = G(x)^{1/2}\left(I + \sum_{i=1}^{m} v_i^N G(x)^{-1/2} G_i G(x)^{-1/2}\right) G(x)^{1/2} > 0$$

$$F(x^+) = F(x)^{1/2}\left(I + \sum_{i=1}^{m} v_i^N F(x)^{-1/2} F_i F(x)^{-1/2}\right) F(x)^{1/2} > 0.$$

This proves the first part of the theorem (x^+ is strictly feasible).

To prove the second part of the theorem note that equations (63)–(65) are also the optimality conditions for a least-squares problem in U and V

$$\begin{array}{ll} \text{minimize} & t\left\|I - G(x)^{1/2}C_1G(x)^{1/2} - \frac{1}{t}G(x)^{1/2}UG(x)^{1/2}\right\|_F^2 \\ & + \left\|I - F(x)^{1/2}C_2F(x)^{1/2} - F(x)^{1/2}VF(x)^{1/2}\right\|_F^2 \\ \text{subject to} & \mathbf{Tr}G_iU + \mathbf{Tr}F_iV = 0, \quad i = 1, \ldots, m. \end{array}$$

The solution is U^N and V^N and the optimal value is λ^2. The same is true for λ^+ if we replace U^N V^N x by $(U^N)^+$ $(V^N)^+$ and x^+ respectively. Therefore

$$\begin{aligned} (\lambda^+)^2 &= t\left\|I - G(x^+)^{1/2}C_1G(x^+)^{1/2} - \frac{1}{t}G(x^+)^{1/2}(U^N)^+G(x^+)^{1/2}\right\|_F^2 \\ &\quad + \left\|I - F(x^+)^{1/2}C_2F(x^+)^{1/2} - F(x^+)^{1/2}(V^N)^+F(x^+)^{1/2}\right\|_F^2 \\ &\leq t\left\|I - G(x^+)^{1/2}C_1G(x^+)^{1/2} - \frac{1}{t}G(x^+)^{1/2}U^NG(x^+)^{1/2}\right\|_F^2 \\ &\quad + \left\|I - F(x^+)^{1/2}C_2F(x^+)^{1/2} - F(x^+)^{1/2}V^NF(x^+)^{1/2}\right\|_F^2 \end{aligned}$$

Without loss of generality we can assume that $G(x^+) = I$ and $F(x^+) = I$. Then

$$\begin{aligned} (\lambda^+)^2 &\leq t\left\|I - C_1 - \frac{1}{t}U^N\right\|_F^2 + \left\|I - C_2 - V^N\right\|_F^2 \\ &= t\left\|I - G(x)^{-1} + \sum_i v_i^N G(x)^{-1}G_iG(x)^{-1}\right\|_F^2 + \left\|I - F(x)^{-1} + \sum_i v_i^N F(x)^{-1}F_iF(x)^{-1}\right\|_F^2 \\ &= t\left\|I - G(x)^{-1} + G(x)^{-2} - G(x)^{-1}\right\|_F^2 + \left\|I - F(x)^{-1} + F(x)^{-2} - F(x)^{-1}\right\|_F^2 \\ &= t\left\|\left(G(x)^{-1} - I\right)^2\right\|_F^2 + \left\|\left(F(x)^{-1} - I\right)^2\right\|_F^2 \\ &\leq \left(t\left\|G(x)^{-1} - I\right\|_F^2 + \left\|F(x)^{-1} - I\right\|_F^2\right)^2 \\ &= \left(t\left\|G(x)^{-1/2}\left(G(x^+) - G(x)\right)G(x)^{-1/2}\right\|_F^2 + \left\|F(x)^{-1/2}\left(F(x^+) - F(x)\right)F(x)^{-1/2}\right\|_F^2\right)^2 \\ &= \lambda^4. \end{aligned}$$

Theorem 5 *If $\lambda < 1$ then*

$$\phi(x) \leq \phi(x^\star) - \lambda - \log(1-\lambda). \tag{71}$$

As a consequenceΓwe have $\phi(x) \leq \phi(x^\star) + \lambda$ if $\lambda \leq 0.81$.

Proof. The problem dual to

$$\begin{array}{ll} \text{minimize} & t\left(c^T x + \log\det G(x)^{-1}\right) + \log\det F(x)^{-1} \\ \text{subject to} & G(x) > 0, F(x) > 0 \end{array} \tag{72}$$

is

$$\begin{array}{ll} \text{maximize} & t\log\det W + \log\det Z - t\mathbf{Tr}G_0 W - \mathbf{Tr}F_0 Z + l + nt \\ \text{subject to} & t\mathbf{Tr}G_i W + \mathbf{Tr}F_i Z = tc_i, \quad i = 1,\ldots,,m \\ & W > 0, \ Z > 0. \end{array} \tag{73}$$

The duality gap associated with an x feasible in (72) and a W and Z feasible in (73) is

$$-t\log\det G(x)W + t\mathbf{Tr}G(x)W - \log\det F(x)Z + \mathbf{Tr}F(x)Z - l - nt.$$

We will establish (71) by showing that the matrices

$$\tilde{W} = C_1 + \frac{1}{t}U^N, \quad \tilde{Z} = C_2 + V^N$$

are dual feasible in (73) with a duality gap less than $-\lambda - \log(1-\lambda)$.

From (63)–(65) it is clear that $\tilde{W}$ and $\tilde{Z}$ satisfy the equality constraint in (73). If $\lambda < 1$ then $\tilde{W}$ and $\tilde{Z}$ are also positive definite. This can be seen from (63) and (64):

$$\begin{aligned} G(x)^{1/2}\tilde{W}G(x)^{1/2} &= I - G(x)^{-1/2}\left(\sum_{i=1}^m v_i^N G_i\right)G(x)^{-1/2} \\ F(x)^{1/2}\tilde{Z}F(x)^{1/2} &= I - F(x)^{-1/2}\left(\sum_{i=1}^m v_i^N F_i\right)F(x)^{-1/2}. \end{aligned}$$

If $\lambda < 1$ then all eigenvalues of $G(x)^{-1/2}\left(\sum_{i=1}^m v_i^N G_i\right)G(x)^{-1/2}$ and $F(x)^{-1/2}\left(\sum_{i=1}^m v_i^N F_i\right)F(x)^{-1/2}$ are less than oneΓand therefore $\tilde{W}$ and $\tilde{Z}$ are positive definite.

We conclude that $\tilde{W}$ and $\tilde{Z}$ are strictly dual feasible in (73). The duality gap associated with $x\Gamma\tilde{W}\Gamma$and $\tilde{Z}$ is

$$\begin{aligned} & -t\log\det G(x)\tilde{W} + t\mathbf{Tr}G(X)\tilde{W} - \log\det F(x)\tilde{Z} + \mathbf{Tr}F(x)\tilde{Z} - l - nt \\ = \ & -t\log\det\left(I - G(x)^{-1/2}\left(\sum_{i=1}^m v_i^N G_i\right)G(x)^{-1/2}\right) - t\mathbf{Tr}G(x)^{-1/2}\left(\sum_{i=1}^m v_i^N G_i\right)G(x)^{-1/2} \\ & - \log\det\left(I - F(x)^{-1/2}\left(\sum_{i=1}^m v_i^N F_i\right)F(x)^{-1/2}\right) - \mathbf{Tr}F(x)^{-1/2}\left(\sum_{i=1}^m v_i^N F_i\right)F(x)^{-1/2} \\ = \ & t\sum_{i=1}^n(-\lambda_i - \log(1-\lambda_i)) + \sum_{i=1}^l(-\gamma_i - \log(1-\gamma_i)) \\ \leq \ & -\lambda - \log(1-\lambda). \end{aligned}$$

The last inequality follows from the power series expansion of $\log(1-x)$ around zero (see also [VB95]).

Number of steps to reach region of quadratic convergence

We first derive an upper bound for $\phi(x+pv)$ as a function of p. No assumption is made on $v\Gamma i.e.\Gamma$it is not necessarily the Newton direction at x.

Let $\lambda_i\Gamma i = 1,\ldots,l\Gamma$be the generalized eigenvalues of $(\sum_i v_i G_i, G(x))\Gamma$and let $\gamma_i\Gamma i = 1,\ldots,n\Gamma$be the generalized eigenvalues of $(\sum_i v_i F_i, F(x))$. Let $\overline{\lambda} = \max(0,\lambda_{\max},\gamma_{\max})\Gamma$and $\underline{\lambda} = \min(0,\lambda_{\min},\gamma_{\min})$. Then $x+pv$ is feasible for $p \in (-\overline{\lambda}^{-1}, -\underline{\lambda}^{-1})$ (if $\underline{\lambda} = 0\Gamma$then the upper bound of the interval is plus infinity; if $\overline{\lambda} = 0\Gamma$then the lower bound of the interval is minus infinity).

We have the following property [NT94].

Theorem 6 *For $p \in (-\overline{\lambda}^{-1}, -\underline{\lambda}^{-1})$,*

$$\frac{1}{(1+p\overline{\lambda})^2}\nabla^2\phi(x) \le \nabla^2\phi(x+pv) \le \frac{1}{(1+p\underline{\lambda})^2}\nabla^2\phi(x). \tag{74}$$

Proof. We need to show that for all $z \in \mathbf{R}^m\Gamma$

$$\frac{1}{(1-p\underline{\lambda})^2}z^T\nabla^2\phi(x)z \le z^T\nabla^2\phi(x-pv)z \le \frac{1}{(1-p\overline{\lambda})^2}z^T\nabla^2\phi(x)z.$$

Let $R_1 \in \mathbf{R}^{l\times l}$ be a matrix that simultaneously diagonalizes $G(x)$ and $\sum_i v_i G_i\Gamma$

$$R_1^T G(x) R_1 = I, \;\; R_1^T\left(\sum_i v_i G_i\right) R_1 = \Lambda,$$

and let $R_2 \in \mathbf{R}^{n\times n}$ be a matrix that simultaneously diagonalizes $F(x)$ and $\sum_i v_i F_i\Gamma$

$$R_2^T F(x) R_2 = I, \;\; R_2^T\left(\sum_i v_i F_i\right) R_2 = \Gamma.$$

As a consequenceΓwe have $R_1^T G(x+pv)R_1 = I + p\Lambda\Gamma$and $R_2^T F(x+pv)R_2 = I + p\Gamma$. Let $\tilde{V} = R_1^T\left(\sum_i z_i G_i\right)R_1\Gamma$and $\tilde{W} = R_2^T\left(\sum_i z_i F_i\right)R_2$. Then

$$\begin{aligned}
& z^T\nabla^2\phi(x+pv)z \\
&= t\mathbf{Tr}\left(\sum_{i=1}^m z_i G_i\right) G(x+pv)^{-1}\left(\sum_{j=1}^m z_j G_j\right) G(x+pv)^{-1} \\
&\quad + \mathbf{Tr}\left(\sum_{i=1}^m z_i F_i\right) F(x+pv)^{-1}\left(\sum_{j=1}^m z_j F_j\right) F(x+pv)^{-1} \\
&= t\mathbf{Tr}R_1^T\left(\sum_{i=1}^m z_i G_i\right) R_1 R_1^{-1} G(x+pv)^{-1} R_1^{-T} R_1^T\left(\sum_{j=1}^m z_i G_i\right) R_1 R_1^{-1} G(x+pv)^{-1} R_1^{-T} \\
&\quad + \mathbf{Tr}R_2^T\left(\sum_{i=1}^m z_i F_i\right) R_2 R_2^{-1} F(x+pv)^{-1} R_2^{-T} R_2^T\left(\sum_{j=1}^m z_j F_j\right) R_2 R_2^{-1} F(x+pv)^{-1} R_2^{-T} \\
&= t\mathbf{Tr}\tilde{W}(I+p\Lambda)^{-1}\tilde{W}(I+p\Lambda)^{-1} + \mathbf{Tr}\tilde{Z}(I+p\Gamma)^{-1}\tilde{Z}(I+p\Gamma)^{-1} \\
&\le \frac{1}{(1+p\underline{\lambda})^2}\left(t\mathbf{Tr}\tilde{W}^2 + \mathbf{Tr}\tilde{Z}^2\right) = \frac{1}{(1+p\underline{\lambda})^2}z^T\nabla^2\phi(x)z.
\end{aligned}$$

This proves the upper bound in the theorem. The lower bound is derived in a similar way.

By integrating the upper bound twice we obtain an upper bound on $f(p) = \phi(x + pv)$

$$f(p) \leq f(0) + pf'(0) + \frac{f''(0)}{\underline{\lambda}^2} \left(p\underline{\lambda} - \log(1 + p\underline{\lambda})\right), \tag{75}$$

assuming $\underline{\lambda} < 0$. The upper bound (75) reaches its minimum for

$$\widehat{p} = \frac{-f'(0)}{\underline{\lambda} f'(0) + f''(0)}.$$

If v is the Newton direction we have $f'(0) = -f''(0)$ and the expressions simplify: $\widehat{p} = 1/(1 - \underline{\lambda})$ and the corresponding upper bound is

$$f(\widehat{p}) \leq f(0) + \frac{f''(0)}{\underline{\lambda}^2} \left(\underline{\lambda} + \log(1 - \underline{\lambda})\right).$$

To obtain an expression in terms of the Newton decrement λ we use the fact that $f''(0) = \lambda^2$ and $0 > \underline{\lambda} \geq -\lambda$ which yields

$$f(\widehat{p}) \leq f(0) - \left(\lambda - \log(1 + \lambda)\right). \tag{76}$$

Proof of Theorem 3

We are now in a position to prove Theorem 3. For simplicity we write $\varphi^{(0)} = \varphi_p(t, x^{(0)})$ and $\varphi^\star = \varphi_p(t, x^\star(t))$. We will prove that the Newton algorithm of §5 terminates within

$$10.7(\varphi^{(0)} - \varphi^\star) + \log_2 \log_2(1/\delta)$$

iterations.

We first consider the initial stage of the algorithm ($\lambda \geq 0.5$). By (76) we know that the function $\phi_p(t, x)$ decreases by at least

$$\lambda - \log(1 + \lambda) \geq 0.5 - \log(1.5) = 0.094$$

at each iteration. The number of iterations required to reach $\lambda \leq 0.5$ is therefore bounded above by

$$(1/0.094)(\varphi^{(0)} - \varphi^\star) \leq 10.7(\varphi^{(0)} - \varphi^\star).$$

During the second stage the algorithm converges quadratically and $\lambda^{(k+1)} \leq (\lambda^{(k)})^2$. Hence $\lambda^{(k)} \leq \delta$ after at most

$$\log_2(\log_2(1/\delta)) + 10.7(\varphi^{(0)} - \varphi^\star)$$

iterations. To complete the proof we note that by Theorem 5

$$\varphi(x) - \varphi^\star \leq -\lambda - \log(1 - \lambda) \leq \lambda$$

if $\lambda \leq 0.5$.