Matlab and Computational Assignments. Please provide a printout of the Matlab code you wrote to generate the solutions to the problems below.

1. **Conjugate Gradient Algorithm.** Recall the linear conjugate gradient algorithm. Download the file [http://users.ece.utexas.edu/~cmcaram/EE381V_2012F/ConjugateGradient.mat](http://users.ece.utexas.edu/~cmcaram/EE381V_2012F/ConjugateGradient.mat). There you will find matrices and vectors defining two equations: $M_1x = b_1$, and $M_2x = b_2$. The solution, $x^*$, is there as well, although this is easy to find since both $M_1$ and $M_2$ are invertible. Use conjugate gradient to solve these two linear systems, and plot the error, $\log(\|x(k) - x^*\|_M^2)$ vs. iteration $k$ for both.

2. **BFGS for Quadratic Problems.** Now use BFGS for the functions above. Note that for non-quadratic problems, as with Newton’s method, BFGS should use an approximate line search. In this quadratic case, however, use an exact line search. In addition to solving and plotting the error, show that BFGS generates conjugate directions. For quadratic problems, BFGS is also guaranteed to terminate in at most $n$ steps.

3. **Newton’s Method.** This problem will demonstrate the two convergence behaviors of Newton’s method, damped and quadratic, by matlab simulation.
Consider $f_m(x) = \|x\|^3 + \frac{m}{2}\|x\|^2$ for $m \in \{0, 0.0001, 0.001, 0.1\}$ and $x \in \mathbb{R}^5$.

(a) For each $m$, implement Newton’s method on $f_m(x)$ and provide the convergence plots, i.e $\log(\|x(k) - x^*\|^2)$ vs. iteration $k$. Use the constant step size $t = 1$.

(b) Using the condition for quadratic convergence, explain how and why your result changes according to $m$.

4. **Central Path.** Consider the linear optimization problem:

$$\begin{align*}
\min : & \quad 2x_1 + 4x_2 + x_3 + x_4 \\
\text{s.t. :} & \quad x_1 + 3x_2 + x_4 \leq 4 \\
& \quad 2x_1 + x_2 \leq 3 \\
& \quad x_2 + 4x_3 + x_4 \leq 3 \\
& \quad x_i \geq 0, \quad i = 1, 2, 3, 4.
\end{align*}$$

(a) Find a function $F$ that is a self-concordant-barrier function, such that the closure of its domain is equal to the feasible set of the problem. (Recall that $-\log(a^T x - b)$ is a self-concordant-barrier function, as you show in an exercise below.)
(b) Find the analytic center $x^*_F$ using Newton’s method. You can initialize at any point in the domain (e.g., $(1/2, 1/2, 1/2, 1/2)$ or any other point you like).

$$x^*_F = \arg\min_{x \in \text{dom}F} F(x).$$

(c) Now you will generate the central path:

$$x^*(t) = \arg\min_{x \in \text{dom}F} f(t; x),$$

where recall:

$$f(t; x) = tc^\top x + F(x).$$

For $t = 0$, the solution, and first point of the central path, is the analytic center. At each iteration, you will compute $t_{k+1} = t_k(1 + \alpha)$. Experiment with different values of $\alpha$. If $\alpha$ is too small, progress may not be that fast as $t$ will grow slowly. If $\alpha$ is too big, we might move outside the region of quadratic convergence, and although $t$ will grow more quickly, each individual step of the central path will take longer to compute.

(d) Plot the error, $\log(||x^{(k)} - x^*||)$ as a function of number of iterations, for different values of $\alpha$.

Note: Chapter 11 in Boyd & Vandenberghe has much information about central path and barrier methods, although the chapter also contains a lot of information, definitions and ideas we have not yet discussed.

5. Do the same for the (slightly larger) LP contained in http://users.ece.utexas.edu/~cmcaram/EE381V_2012F/LP_centralpath.mat In that file you will find specified: $c$, $A$, and $b$, thus defining the problem:

$$\min : \quad c^\top x$$

$$\text{s.t.} : \quad Ax \leq b$$

$$\quad x \geq 0.$$  

Note that you can use CVX to quickly solve both this LP and the previous problem, in order to have the solution.

6. Gradient, Conjugate Gradient, Newton and BFGS. Consider the function (called the Rosenbrock function)

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$  

This function is not convex, however it not hard to see that it has a unique minimizer $x^* = (1, 1)$, and that in a neighborhood of this point, the Hessian is positive definite. Initializing at $x_{\text{init}} = (-1.2, 1)$, implement (a) gradient descent, (b) (non-linear) Conjugate Gradient, (c) BFGS, and (d) Newton, using a back-tracking line search for all four. Plot the error in each as a function of the iteration.

Written Problems

1. We proved quadratic convergence of Newton’s method (locally), using the assumption that $f$ is strongly convex, smooth, and with $L$-Lipschitz Hessian. Locally (so, only after BTLS is taking full steps) derive a rate of convergence in the case where the Hessian is $\alpha$-Hölder continuous. (Recall that a function $F$ is called $\alpha$-Hölder continuous if $||F(x) - F(y)|| \leq H||x - y||^\alpha$ for all $x, y$.)
2. **Self Concordant Barriers.** Recall that a self-concordant function $f$ is called a *self-concordant barrier* if in addition to the properties for self-concordance, it satisfies the property that $\lambda_f(x)^2$ is uniformly bounded by some constant $\nu$. That is, $f$ is called a $\nu$-self-concordant-barrier if

$$\lambda_f(x)^2 = \|\nabla f(x)\|^2 \leq \nu, \quad x \in \text{dom}(f).$$

(a) Explain why we need this property (in connection to minimizing a linear function over a convex set).

(b) For $A$ a positive semidefinite matrix, consider the concave quadratic $\phi(x) = c + b^\top x - \frac{1}{2}x^\top Ax$. Show that $f(x) = -\ln \phi(x)$ is a $\nu$-self-concordant barrier function, with $\nu = 1$. 