The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381V-11: Large Scale Optimization — Fall 2012

PROBLEM SET SEVEN

Caramanis/Sanghavi Due: Thursday, November 8, 2012.

Written Problems

1. Show that sub gradients have the following properties:[1]
   (a) $\partial(\alpha f(x)) = \alpha \partial f(x)$.
   (b) $\partial(f_1 + f_2) = \partial f_1 + \partial f_2$.
   (c) If $g(x) = f(Ax + b)$, then $\partial g(x) = A^\top \partial f(Ax + b)$.
   (d) If $f(x) = \max_{1 \le i \le m} f_i(x)$, then
   $$\partial f(x) = \mathrm{conv} \bigcup_i \{\partial f_i(x),\ f_i(x) = f(x)\}.$$

2. We did this problem in class, but quite quickly. Here's another shot: Compute the sub gradient of the ℓ_1-norm:
   $$f(x) = \|x\|_1 = \sum_i |x_i|.$$

3. Compute the sub gradient of the $\|\cdot\|_{2,1}$ norm on matrices: For M a matrix with columns M_i, this is defined as:
   $$\|M\|_{2,1} = \sum_i \|M_i\|_2.$$

4. (more tricky) Suppose $A_0, A_1, \ldots, A_m$ are symmetric matrices. Consider the function
   $$f(x) = \lambda_{\max}(A(x)),$$
   where
   $$A(x) = A_0 + x_1 A_1 + \cdots + x_m A_m.$$
   Compute the sub gradient of $f(x)$. Hint: use the fact that
   $$f(x) = \sup_{\|y\|_2 = 1} y^\top A(x) y,$$
   and the last property you proved from the first problem.

5. The indicator function of a set $\mathcal{X}$ is denied as:
   $$I_{\mathcal{X}}(x) = \begin{cases} 0 & x \in \mathcal{X} \\ +\infty & \text{otherwise.} \end{cases}$$

[1] We don't need convexity to define sub gradients, but we did use it in the definitions we gave in class. Therefore, in the problems below, you can assume that all functions are convex, coefficients nonnegative, etc.

(a) Show that the sub differential of $I_{\mathcal{X}}$ is the *normal cone* to $\mathcal{X}$ at the point x.

(b) Now consider the constrained optimization problem:

$$\begin{aligned} \text{min}: &\quad f(x) \\ \text{s.t.}: &\quad x \in \mathcal{X}, \end{aligned}$$

for f and $\mathcal{X}$ convex. This can be rewritten as the equivalent unconstrained problem:

$$\text{min}: \; f(x) + I_{\mathcal{X}}(x).$$

This is again a convex function. Write down conditions for a point x^* to be optimal to the unconstrained problem.

6. (Monotonicity). Show that the subdifferential of a convex function $f(\cdot)$ is an example of what is known as a *monotone operator*. That is, show that

$$\langle u - v, x - y \rangle \geq 0, \quad \forall u \in \partial f(x),\; v \in \partial f(y).$$

7. Compute the Legendre-Fenchel Transform of the following:

(a) Quadratic function

$$f(x) = \frac{1}{2} x^\top Q x,$$

where Q is positive definite.

(b) Negative logarithm.

$$f(x) = -\log x.$$

(c) Norm.

$$f(x) = \|x\|,$$

for some norm $\|\cdot\|$.