Matrix Perturbation

Friday, November 08, 2013
2:31 PM

PART ONE

Thm: Let X ind. columns

$$X = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$\mathcal{X} = R(X)$$

$\mathcal{X}$ is A-invariant iff

$$Y^H A X = 0$$

where Y's columns span $\mathcal{X}^\perp$

pf: $\mathcal{X}$ A-invariant

$$\Longleftrightarrow$$

$$A\mathcal{X} \subseteq \mathcal{X}$$

$$\Longleftrightarrow$$

$$A\mathcal{X} \perp \mathcal{X}^\perp$$

$$\Longleftrightarrow$$

$$R(AX) \perp R(Y)$$

$$\Longleftrightarrow$$

$$Y^H A X = 0$$

Matrix Perturbation

Saturday, November 09, 2013
8:53 AM

$\mathcal{X}$ is A-invariant.

X_1 be o.n.b. for $\mathcal{X}$

(X_1, Y_2) be unitary (so columns of (X_1, Y_2) are onb for entire space)

Then:

$$(X, Y_2)^H A (X_1 \ Y_2) = \begin{bmatrix} L_1 & H \\ 0 & L_2 \end{bmatrix}$$

H some matrix

$$\boxed{AX_1 = X_1 L_1}$$

"L_1 is a representation of A on $\mathcal{X} = \mathcal{R}(X_1)$, w.r.t. the basis X_1"

Def'n: $\mathcal{X}$ is a simple invariant subspace

1. invariant (w.r.t. A)
2. $\underbrace{\lambda(L_1)}_{\text{e-values of } L_1} \cap \lambda(L_2) = \phi$

Matrix Perturbation

Saturday, November 09, 2013
8:56 AM

Given: A, B (for us, will be using L_1, L_2)

define: $T_{A,B}(X) = AX - XB$

Thm: T is non-singular iff

$$\lambda(A) \cap \lambda(B) = \phi$$

[in class, proved one direction]

Corollary: $\lambda(T) = \underbrace{\lambda(A) - \lambda(B)}_{\text{elementwise}}$

Exercise: T is a linear operator.

How would we represent "T" as multiplication by a matrix?

→ Any linear operator in finite dims, can be rep'd as v mult'n by a matrix. left

Matrix Perturbation

Saturday, November 09, 2013
8:59 AM

Spectral Resolution

Thm: If $\mathcal{X}$ is A-invariant & simple, then $\mathcal{X}^{\perp}$ is A-invariant,

Exercise: Find an example of an invariant subspace $\mathcal{X}$ where $\mathcal{X}^{\perp}$ is not invariant.

In particular, $\exists$ similarity transformation $\longrightarrow$ may not be unitary! — s.t.

$$(X_1 \ \ X_2)^{-1} A \ (X_1 \ \ X_2) = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$$

$\exists$ X_2, Y_1 s.t.

$$(X_1 \ \ X_2)^{-1} = (Y_1 \ \ Y_2)^H$$

Def'n: $A = X_1 L_1 Y_1^H + X_2 L_2 Y_2^H$

Matrix Perturbation

Saturday, November 09, 2013

9:05 AM

$$\begin{cases} P_1 = X_1 Y_1^H \\ P_2 = X_2 Y_2^H \end{cases}$$

X_i, Y_i play the roll of left and right e-vectors

<u>Exercise</u>: What happens in 1-dim case, i.e., check what happens when $\mathcal{X}$ is a simple invariant 1-dim'l subspace.

<u>Exercise</u>:
- $P_i^2 = P_i \quad i = 1, 2$
- $P_1 P_2 = P_2 P_1 = 0$
- $A = P_1 A P_1 + P_2 A P_2$

P_1 is called the <u>spectral projection</u> of $\mathcal{X}_1$.

We will see in PS exercise, that $\|P_1\|$ controls sensitivity of eigenvalues in L,

Will look at 1-d case.

Matrix Perturbation

Saturday, November 09, 2013

9:58 AM

PART TWO

Last Time

1. Simple invariant subspaces

2. $T : X \longrightarrow AX - XB$

$$(X_1, Y_2)^H A (X_1, Y_2) = \begin{pmatrix} L_1 & H \\ 0 & L_2 \end{pmatrix}$$

"How close are L_1 & L_2"

$\mathcal{X} = \mathcal{R}(X_1)$ is called <u>simple</u> if $\mathcal{L}(L_1) \cap \mathcal{L}(L_2) = \phi$

$$T_{L_1, L_2}(X) = L_1 X - X L_1$$

if T is non-singular

Then $\mathcal{L}(L_1) \cap \mathcal{L}(L_2) = \phi$

and conversely.

Matrix Perturbation

Saturday, November 09, 2013
10:02 AM

Approximation Problem

A, let X_1 be o.n.b.

for an approximate A-invariant

subspace.

$$L = X_1^H A X_1$$

$$\boxed{\text{Recall: If } \mathcal{X} = \mathcal{R}(X_1) \text{ were } A\text{-inv., then, } A X_1 = X_1 L_1}$$

$$R = A X_1 - X_1 L_1$$

$\neq 0$ b/c X_1 is not (quite) A-invariant.

Q: How close is $\mathcal{R}(X_1) = \mathcal{X}$ to being A-invariant, as a function of $\|R\|$

Matrix Perturbation

Saturday, November 09, 2013
10:05 AM

X_1 onb for $\mathcal{X}$

$$(X_1, Y_2)^H A (X_1, Y_2) = \begin{pmatrix} L_1 & H \\ G & L_2 \end{pmatrix}$$

Recall: if $G \equiv 0 \iff \mathcal{X} = R(X_1)$ is A-invariant.

$$G = Y_2^H A X_1$$

Want to find A-invariant subspace "near" $\mathcal{X}$.

$$\hat{X}_1 = (X_1 + Y_2 P)(I + P^H P)^{-1/2}$$

$$\hat{Y}_2 = (Y_2 - X_1 P^H)(I + P^H P)^{-1/2}$$

Exercise: $(\hat{X}_1 \; \hat{Y}_2)$ unitary (inherits from $(X_1 \; Y_2)$)

want to choose: P so that $R(X_1)$ is A-invariant

$R(\hat{X}_1)$ is A-invariant iff

$$\hat{Y}_2^H A \hat{X}_1 = 0$$

Matrix Perturbation

Saturday, November 09, 2013

12:15 PM

$$\hat{Y}_2^H A \hat{X}_1 = 0 \iff$$

$$PL_1 - L_2 P - G + PHP = 0$$

$$\iff \underbrace{PL_1 - L_2 P}_{} = G - PHP$$

$$T_{L_1, L_2}(P) = G - PHP$$

P makes $R(\hat{X}_1)$ A-invariant exactly when it solves the non-linear equation:

$$\boxed{T(P) = G - PHP}$$

Matrix Perturbation

Saturday, November 09, 2013
12:17 PM

$$(X_1\ Y_2)^H A (X_1\ Y_2) = \begin{bmatrix} L_1 & H \\ G & L_2 \end{bmatrix}$$

Thm: Assume that $\lambda(L_1) \cap \lambda(L_2) = \phi$
(implies T is non-singular)

[want: conditions on magnitude of H, G that tell us how far we are from invariant subspace]

Norms used here are "consistent"

$\gamma = \|G\|$

$\eta = \|H\|$

$$\delta = \operatorname{sep}(L_1, L_2) \triangleq \inf_{\|Q\|=1} \|T(Q)\| > 0$$

I. If $\boxed{\dfrac{\gamma\eta}{\delta^2} < 1/4}$ then $\exists\, P$ st $\hat{X}_1$ is A-invariant

II. And, $\|P\| \leq 2\dfrac{\gamma}{\delta}$

in fact: $\|P\| \leq \dfrac{2\gamma}{\delta + \sqrt{\delta^2 - 4\gamma\eta}} \leq 2\dfrac{\gamma}{\delta}$

III. $\hat{X}_1, \hat{Y}_2$ span for A-invariant simple subspaces

$$\hat{L}_1 = (I + P^H P)^{1/2}(L_1 + HP)(I + P^H P)^{-1/2}$$

$$\hat{L}_2 = (I + P^H P)^{1/2}(L_2 - HP)(I + PP^H)^{1/2}$$

Matrix Perturbation

Saturday, November 09, 2013
12:33 PM

Recap + How to use

$$(X_1\ Y_2)^H A\ (X_1\ Y_2) = \begin{pmatrix} L_1 & H \\ G & L_2 \end{pmatrix}$$

Thm says: if $\frac{\gamma\eta}{\delta^2} < 1/4$, $\exists P$, $\|P\| \le 2\frac{\gamma}{\delta}$

s.t. $\hat{X}_1 = (X_1 + Y_2 P)(I + P^H P)^{-1/2}$

$\hat{Y}_2 = (Y_2 - X_1 P^H)(I + P^H P)^{-1/2}$

$R(\hat{X}_1)$ is A-invariant

Q: How close is $\mathcal{X} = R(X_1)$ to $R(\hat{X}_1)$

Distance b/w subspaces: Canonical Angles

$Y_2^H \hat{X}_1 \longrightarrow$ singular values of this matrix are the sines of the Canonical Angles.

$$Y_2^H \hat{X}_1 = P\,(I + P^H P)^{-1/2}$$

If P has singular values: $\pi_1, \pi_2, \ldots$

then $\sin\theta_i = \frac{\pi_i}{\sqrt{1+\pi_i^2}} \le \pi_i$

Canonical angles b/w $R(X_1)$, $R(\hat{X}_1)$

Thm tells us:

$$\|P\| \le 2(\gamma/\delta)$$

Matrix Perturbation

Saturday, November 09, 2013
12:40 PM

P_1 = projection onto $R(X_1)$

$\hat{P}_1$ = " " $R(\hat{X}_1)$

$\|P_1 - \hat{P}_1\|$

Fact: $\|P_1 - \hat{P}_1\|_2 = \sin\theta_1 \leq \pi_1 = \|P\|_2$

operator norm

If $\gamma = \|G\|_2$, $\eta = \|H\|_2$

$$\delta = sep(L_1, L_2) = \inf_{\|Q\|_2 = 1} \|T(Q)\|_2$$

Then $\|P\|_2 \leq 2\frac{\gamma}{\delta}$

$$\Rightarrow \|P_1 - \hat{P}_1\|_2 \leq 2\frac{\gamma}{\delta}.$$

Matrix Perturbation

Saturday, November 09, 2013
12:44 PM

What's left:

① Proof of the theorem

② How to apply the approx'n thm to Matrix Perturbation.

③ What happens & how do things simplify when A is Hermitian.

Matrix Perturbation

Sunday, November 10, 2013

12:38 PM

PART THREE

① Perturbation Thm for general matrices

② Case of Hermitian

→ $A+E$ have to check that this has an invariant subspace

→ Easier, b/c we do not need to check if $\exists$ inv. subspace.

Thm: A matrix

$(X_1\ Y_2)$ $\mathcal{X} = \mathcal{R}(X_1)$ is A-invariant.

$$(X_1, Y_2)^H A (X_1, Y_2) = \begin{pmatrix} L_1 & H \\ 0 & L_2 \end{pmatrix}$$

invariance $\Longleftrightarrow$ 0

Consider: $A+E$,

$$(X_1, Y_2)^H E (X_1, Y_2) = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$$

when Does $A+E$ have an invariant subspace that is close to $\mathcal{X} = \mathcal{R}(X_1)$, and how close?

Matrix Perturbation

Sunday, November 10, 2013
12:46 PM

$$(X_1, Y_2)^H (A+E)(X_1, Y_2) = \begin{bmatrix} L_1 + E_{11} & (H + E_{12}) \\ E_{21} & L_2 + E_{22} \end{bmatrix}$$

H; "G"

From previous (approx'n) theorem:

Let $\|\cdot\| = \|\cdot\|_2$

$$\boxed{\tilde{\gamma} = \|E_{21}\|} \checkmark$$

$$\tilde{\eta} = \|H + E_{12}\| \le \boxed{\|H\| + \|E_{12}\|} \checkmark$$

$$\tilde{\delta} = \text{sep}(L_1 + E_{11}, L_2 + E_{22}) \Rightarrow \text{This expression not ideal.}$$

The theorem then promises, that

if $\dfrac{\tilde{\gamma}\tilde{\eta}}{\tilde{\delta}^2} < 1/4$, then $\exists$ inv-subspace of $A+E$,

in particular, $\exists\, P$, w/ $\|P\|_2 \le 2\dfrac{\tilde{\gamma}}{\tilde{\delta}}$

As we saw when discussing how to use the approx'n thm,

$$\tan \theta_{max} \le \|P\|_2 = 2\frac{\tilde{\gamma}}{\tilde{\delta}}$$

pf: immediate.

Matrix Perturbation

Sunday, November 10, 2013
12:53 PM

Issue: Results we just gave depend on

$\tilde{\delta} \triangleq \mathrm{sep}(L_1 + E_{11}, L_2 + E_{22})$

but ideally we want $\tilde{\delta}$ that depends

on A (L_1, L_2), and on size

of E $(\|E_{11}\|, \|E_{22}\|)$.

Need to better understand: $\boxed{\mathrm{sep}}$

We need the following continuity thm for sep –

Thm: $\mathrm{sep}(L,M) - \|E\| - \|F\| \le \mathrm{sep}(L+E, M+F) \le \mathrm{sep}(L,M) + \|E\| + \|F\|$

pf: Exercise.

Matrix Perturbation

Sunday, November 10, 2013
12:56 PM

Thm: $(X_1\ Y_2)^H A\ (X_1\ Y_2) = \begin{pmatrix} L_1 & H \\ 0 & L_2 \end{pmatrix}$

If E perturbation w/

$$(X_1\ Y_2)^H E\ (X_1\ Y_2) = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$$

Then writing: $\tilde{\gamma} = \|E_{21}\|$

$$\tilde{\eta} = \|H\| + \|E_{12}\|$$

$$\tilde{\delta} = \mathrm{sep}(L_1, L_2) - (\|E_{11}\| + \|E_{22}\|)$$

Then, if $\dfrac{\tilde{\gamma}\tilde{\eta}}{\tilde{\delta}^2} < 1/4$, $\exists$ $(A+E)$-inv. subspace such that

$$\tan \theta_{max} \leq 2\tilde{\gamma}/\tilde{\delta}$$

Summary: Perturbation results depend on:

1. Size of perturbation
2. Matrix A / invariant subspace $\mathcal{X}$

through $T_{L_1,L_2}: X \to L_1 X - X L_2$

$$\inf_{\|Q\|=1} \|T(Q)\| \triangleq \mathrm{sep}(L_1, L_2)$$

Matrix Perturbation

Sunday, November 10, 2013

1:04 PM

Thm: For any square matrices L_1, L_2

$$\inf_{\|Q\|=1} \|T(Q)\| \doteq \operatorname{sep}(L_1, L_2) \le \underbrace{\min |\lambda(L_1) - \lambda(L_2)|}$$

pf: If $\operatorname{sep}(L_1, L_2) = 0$

$\iff$ T singular

$\iff$ $\lambda(L_1) \cap \lambda(L_2) \neq \phi$

If $\operatorname{sep}(L_1, L_2) > 0 \Rightarrow T$ non-singular

Exercise: $\operatorname{sep}^{-1}(L_1, L_2) = \sup_{\|Q\|=1} \|T^{-1}(Q)\| = \|T^{-1}\|$

Exercise: For any consistent norm,

$$\rho(M) \le \|M\|$$

$$\Rightarrow \rho(T^{-1}) \le \|T^{-1}\|$$

Also have shown: $\lambda(T) = \lambda(L_1) - \lambda(L_2)$

$$\Rightarrow \underbrace{\operatorname{sep}^{-1}(L_1, L_2) = \|T^{-1}\| \ge \rho(T^{-1}) = \max |\lambda(L_1) - \lambda(L_2)|^{-1}}$$

This is the statement of the Thm.

Matrix Perturbation

Sunday, November 10, 2013
1:08 PM

Says: If L_1, L_2 have e-values that are close

$\Rightarrow$ sep (L_1, L_2) is small

$\Rightarrow$ $1/\delta$ big

$\Rightarrow$ Perturbation bound large

For Hermitian matrices: the condition that e-values of L_1, L_2 far apart is necessary & sufficient to guarantee stability.

Exercise: For non-H, it's possible for sep(L_1, L_2) to be small but to have e-values of L_1, L_2 well-separated

Compute sep(L_1, L_2) & $\min |\lambda(L_1) - \lambda(L_2)|$

for $L_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $L_2 = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$.

Matrix Perturbation
Sunday, November 10, 2013
1:13 PM

PART FOUR

Perturbation results for Hermitian Matrices

<u>Easier</u>: M is Hermitian, then there $\exists$ a o.n.b. of e-vectors

$M = A$, $M = A + E$

$\Rightarrow$ Always have invariant subspaces.

So can focus on computing bounds — no longer need extra conditions to guarantee existence of inv. subspace.

<u>"Direct Bounds"</u> $\rightarrow$ (Davis & Kahan)

<u>Thm</u>: If L, M Hermitian, then

$$\operatorname{sep}_F(L, M) \triangleq \inf_{\|Q\|_F = 1} \|T(Q)\|_F$$

$$= \min |\lambda(L) - \lambda(M)|$$

Matrix Perturbation

Sunday, November 10, 2013
1:21 PM

<u>Thm</u> : A Hermitian,

$$\underbrace{(X_1\ X_2)}_{\text{unitary}}^H A\,(X_1\ X_2) = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$$

$\mathcal{X} = \mathcal{R}(X_1)$ is A-invariant.

Suppose $\underbrace{Z}_{\text{o.n. columns}}$ same dim'n as X_1

X_1, Z k columns

How close is $\mathcal{R}(Z)$ to $\mathcal{R}(X_1)$

Let M be any $k \times k$ matrix

M is an <u>approximate</u> rep'n of A on Z.

<u>Recall</u>: X_1 is A invariant

$$AX_1 = X_1 L$$

rep'n of A on $\mathcal{X}$, w.r.t. basis X_1

$$(AX_1 - X_1 L = 0)$$

<u>Approximate</u> : $AZ - ZM = R$

Q: How close is $\mathcal{R}(X_1)$, $\mathcal{R}(Z)$?

Matrix Perturbation

Sunday, November 10, 2013
1:27 PM

Note: This is like defining a perturbation $E = \begin{pmatrix} M - L_1 & 0 \\ 0 & 0 \end{pmatrix}$

$$(X_1 \;\; X_2)^H A (X_1 \;\; X_2) = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$$

$$AZ - ZM = R$$

same dim as X

Let $\delta = \min |\lambda(L_2) - \lambda(M)| > 0$

Then: $$\underbrace{\| \sin \Theta [R(X_1), R(Z)] \|_F}_{\text{diagonal matrix with canonical angles b/w } R(X_1),\, R(Z) \text{ on diagonal}} \le \frac{\|R\|_F}{\delta}$$

diagonal matrix with canonical angles b/w $R(X_1)$, $R(Z)$ on diagonal

Matrix Perturbation

Sunday, November 10, 2013
1:30 PM

$$(X_1\ X_2)^H A (X_1\ X_2) = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$$

pf: $R = AZ - ZM$

$$\Rightarrow X_2^H R = X_2^H A Z - X_2^H Z M$$

need Hermitian

$$= L_2 X_2^H Z - X_2^H Z M$$

$$= L_2 (X_2^H Z) - (X_2^H Z) M$$

$$= T_{L_2,M} (X_2^H Z)$$

Now: $\delta = \mathrm{sep}_F (L_2, M)$

$$\triangleq \inf_{\|Q\|_F = 1} \|T(Q)\|_F$$

$$\Rightarrow \|T(Q)\|_F \geq \delta \cdot \|Q\|_F \quad \forall Q$$

$$Q = X_2^H Z$$

$$\delta \|X_2^H Z\|_F \leq \|T(Q)\|_F$$

$$\Rightarrow \|X_2^H Z\|_F \leq \frac{\|R\|_F}{\delta}$$

$R(X_1)$, $R(Z)$

$$\|X_2^H Z\|_F = \boxed{\|\sin\Theta[R(X_1), R(Z)]\|_F \leq \frac{\|R\|_F}{\delta}}$$

Matrix Perturbation

Sunday, November 10, 2013

1:37 PM

$$R = AZ - ZM$$

Then:
$$\| \sin \Theta [R(X_1), R(Z)] \|_F \le \frac{\|R\|_F}{\delta}$$

$$\boxed{\delta = \text{sep}_F = \min | \lambda(L_2) - \lambda(M) |}$$

When is this useful?

For op-norm bound this assumption not enough.

If dim of X_1 (Z) is small,

this result can give useful bounds.

But, if dim of X_1 is large, then

the Frobenius norm becomes large

If dim $X_1 = 1$ then Frob. norm = Op norm.

But for large dimensional invariant subspaces:

Would like results in terms of operator norm not Frobenius norm.

For this: we need further restrictions on exactly how the e-values of L_2, M are separated.

Matrix Perturbation

Sunday, November 10, 2013

1:41 PM

Lemma: Let $\|\cdot\|$ be a consistent norm, with

$$\|A\| \le \alpha$$
$$\|B^{-1}\|^{-1} \ge \alpha + \delta \quad \text{for } \delta > 0$$

If $AX - XB = C$

Then $$\|X\| \le \frac{\|C\|}{\delta}$$

pf: By consistency: $\|AX\| \le \|A\| \cdot \|X\| = \alpha \|X\|$

$\|XB\| \ge (\alpha + \delta)\|X\|$

$\Rightarrow \quad C = AX - XB$

$\|C\| \ge \|BX\| - \|AX\| \ge \delta \|X\|$

How will we use this:

In prev. thm: $R = AZ - ZM$

Here: need to translate the Lemma's assumptions on $\|A\|$, $\|B^{-1}\|^{-1}$ into e-value conditions.

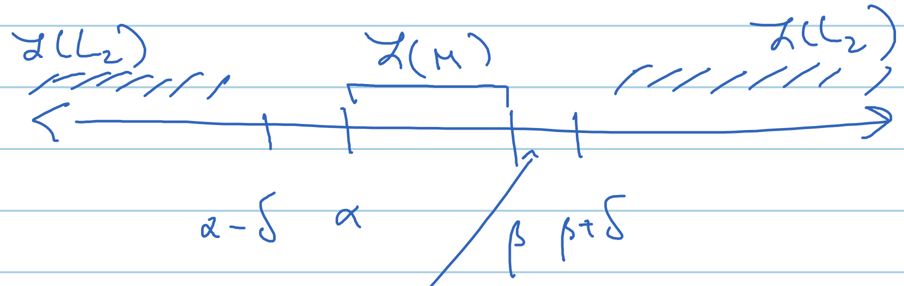
Matrix Perturbation

Sunday, November 10, 2013
1:45 PM

Thm: In the same context as in the previous sin-Θ theorem

If $\lambda(M) \subseteq [\alpha, \beta]$ and for $\delta > 0$

$$\lambda(L_2) \subseteq \mathbb{R} \setminus [\alpha - \delta, \beta + \delta]$$



$$\delta \leq \min |\lambda(L_2) - \lambda(M)|$$

but this

but the above interval condition represents additional assumption

Then for any unitarily invariant norm $(\|\cdot\|_{op} = \|\cdot\|_2, \ \|\cdot\|_F)$

$$\|\sin \Theta [R(X_1), R(Z)]\| \leq \frac{\|R\|}{\delta}$$

If $\|\cdot\| = \|\cdot\|_2$ then this Thm says

$$\boxed{\sin \theta_{max} \leq \frac{\|R\|}{\delta}}$$

Matrix Perturbation

Sunday, November 10, 2013
1:50 PM

pf: Check that WLOG we can assume that $\alpha = -\beta$

and $\mathcal{L}(M) \subseteq [-\alpha, \alpha]$

$\mathcal{L}(L_2) \subseteq (-\infty, -\alpha-\delta) \cup (\alpha+\delta, \infty)$

To apply previous lemma:

$\|M\| \leq \alpha$, $\|L_2^{-1}\|^{-1} \geq \alpha + \delta$

Similarly to proof of previous $\sin\Theta$ theorem,

since

$$(X_1 \ X_2)^H A (X_1 \ X_2) = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$$

and $R = AZ - ZM$

$$\Rightarrow X_2^H R = X_2^H A Z - X_2^H Z M$$
$$= L_2 (X_2^H Z) - (X_2^H Z) M$$

Now we can apply the lemma above using the operator norm:

unitarily invariant norm.

$$\|X_2^H Z\| \leq \frac{\|X_2^H R\|}{\delta} = \frac{\|R\|}{\delta}$$

In other words:

$$\|\sin\Theta[R(X_1), R(Z)]\| \leq \frac{\|R\|}{\delta}$$

Matrix Perturbation

Sunday, November 10, 2013
1:55 PM

Last two results: stated in terms of any Z, M.

Let's apply above sin-Θ thm directly to perturbation:

Thm: M, $M+\Delta$ Hermitian

hence we can write: $M = (X_1\ X_2)^H \begin{bmatrix} M_0 & \\ & M_1 \end{bmatrix} (X_1\ X_2)$

$$M+\Delta = (Y_1\ Y_2)^H \begin{bmatrix} \hat{M}_0 & \\ & \hat{M}_1 \end{bmatrix} (Y_1\ Y_2)$$

where $\|\Delta\|$ is given

$\mathcal{L}(M_0) \subseteq [a,b]$, $\mathcal{L}(\hat{M}_1) \subseteq (-\infty, a-\delta) \cup (b+\delta, \infty)$

$\mathcal{X} = R(X_1)$ M-invariant

$\mathcal{Y} = R(Y_1)$ $(M+\Delta)$-invariant

Let $R = MY_1 - Y_1\hat{M}_0$

$$\|R\| = \|MY_1 - Y_1\hat{M}_0\| = \|(M+\Delta)Y_1 - Y_1\hat{M}_0 - Y_1\Delta\|$$

$$\le \|(M+\Delta)Y_1 - Y_1\hat{M}_0\| + \|Y\Delta\|$$

$$= 0 + \|\Delta\|$$

$$\Rightarrow \|\sin\Theta[R(X_1), R(Y_1)]\| \le \frac{\|\Delta\|}{\delta}.$$

$\boxed{\text{pf is }\underline{\text{immediate}}}$