The Technion (and The University of Texas at Austin)
Department of Electrical and Computer Engineering

Large Scale Learning — Fall 2013

Assignment 2

Constantine Caramanis Not Due

There are numerous problems marked as *Exercise* given during the class, that are meant to fill in some missing details. These are not replicated here. The point of the exercises below is more of the same: to provide practice and review, and also to fill in details left out in class.

1. Show that the Frobenius norm and all induced norms are submultiplicative, or otherwise called *consistent*, i.e., for any matrices A and B (of compatible dimension), $\|AB\| \leq \|A\| \cdot \|B\|$.

2. Find an example of a matrix norm that is not submultiplicative. That is, find norm, $\|\cdot\|$ that is indeed a norm, i.e., (i) $A \neq 0$ implies $\|A\| > 0$, (ii) $\|\alpha A\| = |\alpha|\|A\|$, and (iii) $\|A+B\| \leq \|A\| + \|B\|$, and yet the inequality: $\|AB\| \leq \|A\| \cdot \|B\|$ is not always satisfied.

3. Prove the following basic fact that we have been using freely in class: suppose that X_1 is an $n \times k$ matrix whose column form an orthonormal basis for $\mathcal{X}$, and let X_2 be a completion so that (X_1, X_2) form an orthonormal basis for the entire space. Similarly, the columns of Y_1 form an orthonormal basis for $\mathcal{Y}$, and (Y_1, Y_2) is an orthonormal basis for the entire space. Show that the singular values of $X_2^H Y_1$ and $Y_2^H X_1$ are the sines of the canonical angles between $\mathcal{X}$ and $\mathcal{Y}$.

4. For the same setting as above, and for $\|\cdot\|$ denoting the operator norm, prove:

$$\|X_2^H Y_1\| = \|Y_2^H X_1\| = \|\sin\Theta\| = \|X_1 X_1^H - Y_1 Y_1^H\|_2.$$

   (Note that the first two equalities are the conclusion of the previous problem – here you are asked to prove the third).

5. Compute via direct calculation, the perturbation in the subspace spanned by the eigenvector corresponding to the bottom eigenvector, for the following matrix and perturbation, as a function of a and h:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \qquad \Delta = \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}.$$

6. Repeat the above computation using the $\sin\Theta$ theorem developed in class, and compare to what you got above. Note that the matrices above are symmetric, so you should be using the appropriate results. (You will probably want to use the last theorem from part 4 of the video lectures from November 6.)