The Technion (and The University of Texas at Austin)
Department of Electrical and Computer Engineering

Large Scale Learning — Fall 2013

Assignment 3

Constantine Caramanis Not Due

There are numerous problems marked as *Exercise* given during the class, that are meant to fill in some missing details. These are not replicated here. The point of the exercises below is more of the same: to provide practice and review, and also to fill in details left out in class.

1. Matrix Perturbation Review

   (a) Go back to the main $\sin\Theta$ theorem we proved via the so-called *direct method* and prove the following: Suppose $LX - XM = R$, for some consistent (submultiplicative) norm. Then if we have for some $\alpha, \delta > 0$

   $$\|L^{-1}\|^{-1} \geq \alpha + \delta, \quad \text{and} \quad \|M\| \leq \alpha,$$

   or if

   $$\|M^{-1}\|^{-1} \geq \alpha + \delta, \quad \text{and} \quad \|L\| \leq \alpha,$$

   we have

   $$\|X\| \leq \frac{\|R\|}{\delta}.$$

   (b) Consider a Hermitian matrix A and a Hermitian perturbation $A + \Delta$. Suppose that $X = [\ X_1 \quad X_2\ ]$ and $Y = [\ Y_1 \quad Y_2\ ]$ are orthonormal bases (of matching dimensions for X_i and Y_i), such that

   $$X^H A X = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \qquad Y^H (A + \Delta) Y = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

   Then if either M_1 and L_2, or L_1 and M_2 are separated, in the sense that one matrix has spectrum contained inside an interval and the other has spectrum outside a δ-expansion of that same interval, then the sine of the largest canonical angle between subspaces $\mathcal{X} = \text{Range}(X_1)$ and $\mathcal{Y} = \text{Range}(Y_1)$ is bounded above by $\|\Delta\|/\delta$, where $\|\cdot\|$ denotes the spectral norm (equal to the operator norm since everything is Hermitian).

2. The following problems are useful for the matrix sketching / randomized linear algebra that we are covering.

   (a) Given a matrix X with full column rank, let P_X denote the orthogonal projector onto its range, i.e., the self-adjoint operator that satisfies: $P_X(x) = x$ for every $x \in \text{Ra}(X)$, and $P_X(y) = 0$ for every $y \perp \text{Ra}(X)$. Show that:

   $$P_X = X(X^* X)^{-1} X^*.$$

(b) Suppose that a matrix M is positive semi-definite. Show first that $(I + M)$ is invertible, and then the following result on perturbation of inverses.

$$M \succeq I - (I + M)^{-1}.$$

(c) Suppose that M is a positive semidefinite matrix, with block decomposition:

$$M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}.$$

Show that the following holds for $\|\cdot\|$ denoting the operator (spectral) norm:

$$\|M\| \leq \|A\| + \|C\|.$$

(Hint: you can use, or if you like prove, the fact that $M \succeq 0$ implies that $\|B\|^2 \leq \|A\| \cdot \|C\|$, a fact that is immediate for a 2×2 matrix.)