The University of Texas at Austin
Department of Electrical and Computer Engineering

EE362K: Introduction to Automatic Control—Fall 2009

SOLUTIONS TO PROBLEM SET ONE

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1. Left to the students.

2. The Taylor expansions of $\cos(\alpha t)$, $\sin(\alpha t)$, $e^{i\alpha t}$ and $e^{-i\alpha t}$ are:

$$\begin{aligned}\cos(\alpha t) &= 1 - \frac{\alpha^2 t^2}{2!} + \frac{\alpha^4 t^4}{4!} - \dots \\ \sin(\alpha t) &= \alpha t - \frac{\alpha^3 t^3}{3!} + \frac{\alpha^5 t^5}{5!} - \dots \\ e^{i\alpha t} &= \left(1 - \frac{\alpha^2 t^2}{2!} + \frac{\alpha^4 t^4}{4!} - \dots\right) + i\left(\alpha t - \frac{\alpha^3 t^3}{3!} + \frac{\alpha^5 t^5}{5!} - \dots\right) \\ e^{-i\alpha t} &= \left(1 - \frac{\alpha^2 t^2}{2!} + \frac{\alpha^4 t^4}{4!} - \dots\right) - i\left(\alpha t - \frac{\alpha^3 t^3}{3!} + \frac{\alpha^5 t^5}{5!} - \dots\right),\end{aligned}$$

from which one can see that we have the identities:

$$\begin{aligned}\cos(\alpha t) &= \frac{e^{i\alpha t} + e^{-i\alpha t}}{2} \\ \sin(\alpha t) &= \frac{e^{i\alpha t} - e^{-i\alpha t}}{2i},\end{aligned}$$

and also the desired identity:

$$e^{i\alpha t} = \cos(\alpha t) + i\sin(\alpha t).$$

Now consider some $a \in \mathbb{C}$, and write $a = \beta + i\alpha$. Then we have:

$$e^{at} = e^{\beta t + i\alpha t} = e^{\beta t} \cdot e^{i\alpha t} = e^{\beta t}(\cos(\alpha t) + i\sin(\alpha t)).$$

It is clear from this expression that the magnitude of e^{at} depends only on $\beta = Re(a)$.

3. (a) $Av_1 = 0$ and $Av_2 = 0 \Rightarrow Av = A(\alpha v_1 + \beta v_2) = \alpha(Av_1) + \beta(Av_2) = \alpha \times 0 + \beta \times 0 = 0$

   (b) Let $v = [x_1 \quad x_2 \quad x_3]^T$. Then $Av = 0$ gives the following set of equations:

$$\begin{aligned}x_1 + 3x_2 + 4x_3 &= 0 \\ 2x_1 - x_3 &= 0 \\ x_1 + x_2 + 2x_3 &= 0\end{aligned}$$

   Solving gives $x_1 = x_2 = x_3 = 0$ i.e. the set of vectors satisfying $Av = 0$ is $\{[0 \quad 0 \quad 0]^T\}$.

   (c) The determinant of A can be found to be -6.

(d) Let $v = [x_1 \quad x_2 \quad x_3]^T$. Then $Av = 0$ gives the following set of equations:

$$\begin{aligned} x_1 + 3x_2 + 4x_3 &= 0 \\ 2x_1 - x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

Solving gives $x_2 = -3x_1, x_3 = 2x_1$. Fixing $x_1 = \lambda \Rightarrow x_2 = -3\lambda, x_3 = 2\lambda$, where $\lambda \in \mathbf{R}$. So, the set of vectors satisfying $Av = 0$ is $\{[\lambda \ -3\lambda \quad 2\lambda]^T : \lambda \in \mathbf{R}\}$

(e) The determinant of the new matrix A can be found to be 0.

Indeed, this exercise is an illustration of something much more general which we will see, and use extensively later in the class. We will see that the matrix equation $Ax = 0$ has the unique solution $x = 0$ if and only if the determinant of A is different from zero (can be positive or negative). Moreover, a matrix A is invertible if and only if the equation $Ax = 0$ has the unique solution $x = 0$. Therefore, a matrix A is invertible, i.e., there exists some matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$, if and only if the determinant of A (denoted $\det A$) is different from zero.

4. We convert the differential equation

$$q^{(4)} + 3q^{(3)} - 2\ddot{q} + \dot{q} + 2q = 0.$$

to standard form as follows. Define new variables: $x_1 = q^{(3)}$, $x_2 = q^{(2)} = \ddot{q}$, $x_3 = \dot{q}$, and $x_4 = q$. Then rearranging the terms of the equation above, we find:

$$\begin{aligned} \dot{x}_1 &= q^{(4)} = -3q^{(3)} + 2\ddot{q} - \dot{q} - 2q = -3x_1 + 2x_2 - x_3 - 2x_4 \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_2 \\ \dot{x}_4 &= x_3. \end{aligned}$$

Writing this in matrix form, we find:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{bmatrix} -3 & 2 & -1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Using Matlab (or Octave, or some other approach) we can compute the eigenvalues of this matrix.

5. - $x_1 = x, x_2 = y, x_3 = \theta, x_4 = \dot{x}, x_5 = \dot{y}, x_6 = \dot{\theta}$ give:

$$\underbrace{\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{x_5} \\ \dot{x_6} \end{pmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ -g\sin x_3 - \frac{c}{m}x_4 + \frac{u_1}{m}\cos x_3 - \frac{u_2}{m}\sin x_3 \\ g(\cos x_3 - 1) - \frac{c}{m}x_5 + \frac{u_1}{m}\sin x_3 + \frac{u_2}{m}\cos x_3 \\ \frac{r}{J}u_1 \end{pmatrix}}_{f(\mathbf{x},\mathbf{u})}$$

where $\mathbf{x} = (x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6)^T, \mathbf{u} = (u_1 \ u_2)^T$.

- $\mathbf{u} = (0\ 0)^T = \mathbf{0}$ gives:

$$f(\mathbf{x}, \mathbf{0}) = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ -g \sin x_3 - \frac{c}{m} x_4 \\ g(\cos x_3 - 1) - \frac{c}{m} x_5 \\ 0 \end{pmatrix}$$

$(x, y, \theta) = (0, 0, 0) \Rightarrow \dot{x} = \dot{y} = \dot{\theta} = 0 \Rightarrow \mathbf{x} = \mathbf{0}$. By definition, $\mathbf{x} = \mathbf{x_e}$ is an equilibrium for the system if $f(\mathbf{x_e}, \mathbf{0}) = \mathbf{0}$. Since $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$, $\mathbf{x} = \mathbf{0}$ and hence $(x, y, \theta) = (0, 0, 0)$ is an equilibrium point for the system.

The Jacobian of f is given by:

$$\mathbf{J}_{f(\mathbf{x},\mathbf{0})} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g \cos x_3 & -\frac{c}{m} & 0 & 0 \\ 0 & 0 & -g \sin x_3 & 0 & -\frac{c}{m} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$(x, y, \theta) = (0, 0, 0)$ i.e. $\mathbf{x} = \mathbf{0}$ gives:

$$\mathbf{J}_{f(\mathbf{0},\mathbf{0})} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g & -\frac{c}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{c}{m} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

6. Left to the students.