The University of Texas at Austin
Department of Electrical and Computer Engineering

# EE362K: Introduction to Automatic Control—Fall 2009

## Solutions to Problem Set Three

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1. - We use the notation $|A|$ to denote the determinant of a square matrix $A$. The rule for determinant of the product of matrices gives $|T^{-1}AT| = |T||A||T^{-1}|$. Also $TT^{-1} = I$ gives $|T^{-1}| = |T|^{-1}$, therefore, $|T^{-1}AT| = |T||A||T^{-1}| = |T||A||T|^{-1} = |A|$.
   Denoting the characteristic polynomials of $A$ and $T^{-1}AT$ by $p_A(\lambda)$ and $p_{T^{-1}AT}(\lambda)$ gives

$$\begin{aligned} p_A(\lambda) &= |\lambda I - A| = |T|^{-1}|\lambda I - A||T| = |T^{-1}||\lambda I - A||T| \\ &= |T^{-1}(\lambda I - A)T| = |\lambda T^{-1}T - T^{-1}AT| = |\lambda I - T^{-1}AT| \\ &= p_{T^{-1}AT}(\lambda) \end{aligned}$$

   Since the characteristic polynomials of $A$ and $T^{-1}AT$ are same, they have the same eigenvalues as eigenvalues are roots of the characteristic polynomial.

   - For a square matrix $A$, its eigenvalue $\lambda$ and corresponding eigenvector $v$ satisfy $Av = \lambda v$. Since $TT^{-1} = I$, we have $ATT^{-1}v = \lambda v$. Multiplying both sides by $T^{-1}$ from the left gives $T^{-1}ATT^{-1}v = \lambda(T^{-1}v) \Rightarrow (T^{-1}AT)(T^{-1}v) = \lambda(T^{-1}v) \Rightarrow (T^{-1}AT)w = \lambda w$, where $w \triangleq T^{-1}v$. Hence, $w = T^{-1}v$ is the eigenvector corresponding to the eigenvalue $\lambda$ for the matrix $T^{-1}AT$.

2. (a) We assume $\alpha \neq 0$ or else the utility of feedback would be eliminated. Substituting $u = -\alpha x$ in the differential equation and solving for $\dot{x} = 0$ gives

$$\frac{1}{1+x} - 1 - \alpha x = 0 \Rightarrow x\left(x + 1 + \frac{1}{\alpha}\right) = 0 \Rightarrow x = 0, -1 - \frac{1}{\alpha}$$

   Hence, the equilibrium points are $x_e = 0, -1 - \frac{1}{\alpha}$.

   (b) $V(x) = \frac{1}{2}x^2$ is positive definite about $x_e = 0$ as $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$.

$$\dot{V}(x) = x\dot{x} = -\alpha x^2 \left(\frac{x + 1 + \frac{1}{\alpha}}{x+1}\right)$$

   For stability about $x_e = 0$, $\dot{V}(x)$ must be locally negative semi-definite about $x_e = 0$. As $\dot{V}(0) = 0$, we only require $\dot{V}(x) \leq 0$ for $x \neq 0$. Since $x^2 > 0$ for $x \neq 0$, we have

$$\begin{aligned} \dot{V}(x) = -\alpha x^2 \left(\frac{x + 1 + \frac{1}{\alpha}}{x+1}\right) \leq 0 \quad &\Rightarrow \quad \frac{x + 1 + \frac{1}{\alpha}}{x+1} \geq 0 \quad \text{if } \alpha > 0 \\ &\Rightarrow \quad \frac{x + 1 + \frac{1}{\alpha}}{x+1} \leq 0 \quad \text{if } \alpha < 0 \end{aligned}$$

Consider the case $\alpha > 0$; the range of $x$ satisfying the inequality can be found to be $x \geq -1$ or $x \leq -1 - \frac{1}{\alpha}$ i.e. $(-\infty, -1 - \frac{1}{\alpha}] \cup [-1, \infty)$. This range always includes a neighborhood around $x_e = 0$, whatever positive value $\alpha$ takes. So, stability about $x_e = 0$ is achieved for $\alpha > 0$ in this case.

Next consider the case $\alpha < 0$; the range of $x$ satisfying the inequality can be found to be $-1 \leq x \leq -1 - \frac{1}{\alpha}$ i.e. $[-1, -1 - \frac{1}{\alpha}]$. This range includes a neighborhood around $x_e = 0$ only if $-1 - \frac{1}{\alpha} > 0 \Rightarrow \alpha > -1$. So, stability about $x_e = 0$ is achieved for $-1 < \alpha < 0$ in this case.

Combining the results of the two cases gives $\alpha > 0$ or $-1 < \alpha < 0$ i.e. $\alpha > -1, \alpha \neq 0$. Hence, the Lyapunov function shows the system is stable for $\alpha > -1, \alpha \neq 0$.

3. 
- $V_1(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$:
  $V_1(0,0) = 0$ and $V_1(x) > 0$ for $x \neq 0$, so that $V_1(x)$ is positive definite. Next we observe if $\dot{V_1}(x)$ is negative semi-definite.

$$\begin{aligned} \dot{V_1}(x) &= \tfrac{\partial V_1}{\partial x_1}\dot{x_1} + \tfrac{\partial V_1}{\partial x_2}\dot{x_2} \\ &= x_1\dot{x_1} + x_2\dot{x_2} \\ &= -ax_1^2 - bx_1x_2 - cx_2^2. \end{aligned}$$

  $\dot{V_1}(x)$ can be written as $\dot{V_1}(x) = -x^TQx$, where $x = [x_1\, x_2]^T$ and

$$Q \triangleq \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$$

  $\dot{V_1}(x) = -x^TQx$ being negative semi-definite is equivalent to $Q$ being positive semi-definite, so that $Q$ has non-negative eigenvalues. The characteristic polynomial for $Q$ is given by

$$p_Q(\lambda) = \lambda^2 - (a+c)\lambda + ac - \frac{b^2}{4}$$

  The roots of $p_Q(\lambda)$ (i.e. the eigenvalues) are non-negative if their sum $a+c$ and product $ac - \frac{b^2}{4}$ are non-negative. In other words, $a + c \geq 0$ and $b^2 \leq 4ac$.

- $V_2(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - \frac{b}{c-a}x_1)^2$:
  We require $a \neq c$ or else $V_2(x)$ won't be defined. $V_2(0,0) = 0$ and $V_2(x) > 0$ for $x \neq 0$. Next we observe if $\dot{V_2}(x)$ is negative semi-definite.

$$\begin{aligned} \dot{V_2}(x) &= \tfrac{\partial V_2}{\partial x_1}\dot{x_1} + \tfrac{\partial V_2}{\partial x_2}\dot{x_2} \\ &= (x_1 - \tfrac{b}{c-a}(x_2 - \tfrac{b}{c-a}x_1))\dot{x_1} + (x_2 - \tfrac{b}{c-a}x_1)\dot{x_2} \\ &= -(a + \tfrac{ab^2}{(c-a)^2} - \tfrac{b^2}{c-a})x_1^2 - (-\tfrac{2ab}{c-a})x_1x_2 - cx_2^2. \end{aligned}$$

  Arguing like in the previous part gives the conditions

$$a + c + \frac{ab^2}{(c-a)^2} - \frac{b^2}{c-a} \geq 0 \quad \text{and} \quad \frac{a^2b^2}{(c-a)^2} \leq c\left(a + \frac{ab^2}{(c-a)^2} - \frac{b^2}{c-a}\right)$$

  where $a \neq c$.

4. Since $\dot{x} = Ax + Bu$, $y = Cx$ and $u = -ky$, we have $\dot{x} = (A - kBC)x = \bar{A}x$. For the given system we have:

$$\bar{A} = \begin{pmatrix} k & 1 \\ -4k & -3 \end{pmatrix}$$

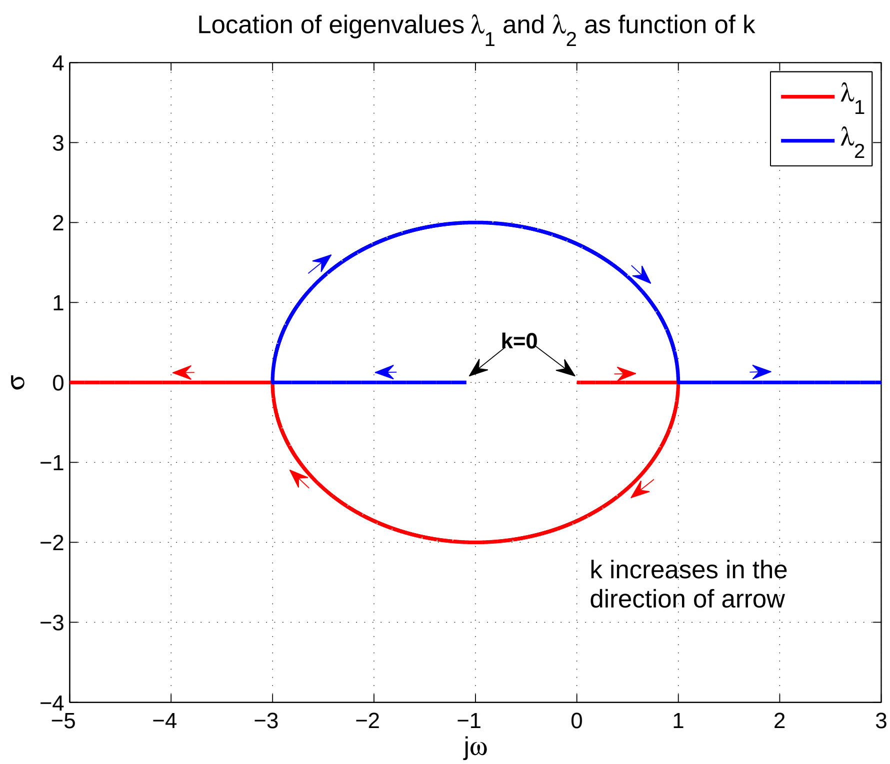
The characteristic polynomial of $\bar{A}$ is

$$p_{\bar{A}}(\lambda) = \lambda^2 + (k-3)\lambda + k$$

The eigenvalues of $\bar{A}$ are

$$\lambda = \frac{3 - k \pm \sqrt{k^2 - 10k + 9}}{2}$$

The following figure depicts the locus of the eigenvalues in the complex plane.




5. (a) We prove by contradiction. For the $i$th and $j$th eigenvalues and their corresponding eigenvectors we have: $Av_i = \lambda_i v_i$ and $Av_j = \lambda_j v_j$ respectively. If $v_i = v_j$ for some $i$ and $j$, then we have $0 = A(v_i - v_i) = A(v_i - v_j) = \lambda_i v_i - \lambda_j v_j = v_i(\lambda_i - \lambda_j)$. Since $\lambda_i \neq \lambda_j$, the only way for this to occur is $v_i = v_j = 0$, which is a contradiction, since eigenvectors are nonzero.

   (b) Since there are $n$ eigenvectors, showing that these form a basis can be done in one of two equivalent ways: showing that the $n$ vectors are linearly independent, or showing that they span the entire space. Here we follow the former approach, and show that the $n$ vectors are independent. We proceed by contradiction. Assume that the eigenvectors are linearly dependent and so that

$$\sum_{i=1}^{n} \alpha_i v_i = 0$$

where not all coefficients $\alpha_i$ are zero. We know that $(\alpha_1, \ldots, \alpha_n)$ is nonzero – but how many nonzero elements does it have? Since we know that each $v_i$ is different from zero, the vector $(\alpha_1, \ldots, \alpha_n)$ must have *at least two* nonzero elements, otherwise the equation above would read: $\alpha_i v_i = 0$, hence $v_i = 0$, which we know is not true. By the result of part (a), we further know that no two eigenvectors are equal. This means that no two eigenvectors can be multiples of each other. In particular, this means that the vector $(\alpha_1, \ldots, \alpha_n)$ must have *at least three* nonzero elements, otherwise the equation above would read $\alpha_i v_i + \alpha_j v_j = 0$, which is equivalent to $v_i = (-\alpha_j/\alpha_i)v_j$.

We'll show that if some vector $(\alpha_1, \ldots, \alpha_n)$ exists satisfying the above condition, then another must exist that satisfies the condition above and has only 2 nonzero elements. This will directly contradict our discussion above, and the contradiction will prove the claim. Let $(\alpha_1, \ldots, \alpha_n)$, be a vector satisfying the equation above, and suppose that $k$ of its elements are nonzero. We will show that we can find another vector with $k-1$ elements nonzero. Repeating this will give us our contradiction.

We have

$$\sum_{i=1}^{n} \alpha_i v_i = 0, \tag{1}$$

Multiplying the matrix $A$ to both sides gives

$$\sum_{i=1}^{n} \alpha_i \lambda_i v_i = 0. \tag{2}$$

We can find some index $i$ so that $\lambda_i$ and $\alpha_i$ are both nonzero. Without any loss of generality, let the index be $i = 1$. Now we multiply (1) by $\lambda_1$, and subtract (2) to get

$$\begin{aligned} 0 &= \lambda_1 \sum_{i=1}^{n} \alpha_i v_i - \sum_{i=1}^{n} \alpha_i \lambda_i v_i \\ &= \sum_{n=1}^{n} (\lambda_1 - \lambda_i)\alpha_i v_i \\ &= 0 + \sum_{n=2}^{n} (\lambda_1 - \lambda_i)\alpha_i v_i. \end{aligned}$$

Consider now the vector $(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$, where $\tilde{\alpha}_j = (\lambda_1 - \lambda_j)\alpha_j$. This new vector has exactly one less non-zero component than our previous vector. Since $\lambda_i \neq \lambda_j$, no other component other than the first was zeroed out, so there is no danger that $(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ is the all-zeros vector. Thus, we constructed a vector with only $k-1$ nonzero components that satisfies

$$\sum \tilde{\alpha}_i v_i = 0.$$

Repeating this will give us our contradiction.

(c) Let $T = [v_1\, v_2 \ldots v_n]$ and denote $\Lambda \triangleq diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $AT = [Av_1\, Av_2 \ldots Av_n] = [\lambda_1 v_1\, \lambda_2 v_2 \ldots \lambda_n v_n] = T\Lambda$. Hence, $T^{-1}AT = \Lambda$.

(d) Assuming all elements of $A$ are real, it can be readily verified that all the coefficients of the characteristic polynomial are real and hence, the complex roots of it appear in complex conjugate pairs. Now, if required, we rearrange rows of $A$ such that the complex conjugate eigenvalues are adjacent. We prove for a single block and extend it for arbitrary number of complex eigenvalues. Consider a matrix in the form of:

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

Observe that the characteristic function of $M$ is the multiplication of the characteristic functions of $M_1$ and $M_2$, i.e. $p_M(\lambda) = p_{M_1}(\lambda)p_{M_2}(\lambda)$. Also, if $T = [v_i]^T$ is the eigenvector matrix corresponding to $M$, it can be written in the block diagonal form as well; i.e.

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

where $T_1$ and $T_2$ are the corresponding eigenvector matrices. This fact can be further generalized to $m$ block matrics. Now we focus on a single complex conjugate pair and show that each diagonal block of complex conjugate eigenvalues can be transformed to a block of the desired form. Let us denote the pair of eigenvalues by $\lambda_{i,i+1} = \sigma \pm j\omega$ now we can verify that the characteristic function of $2 \times 2$ diagonal matrix based on these two eigenvalues is:

$$p_{\Lambda'_i}(\lambda) = (\lambda - \lambda_i)(\lambda - \lambda_{i+1}) = \lambda^2 - 2\sigma\lambda + (\sigma^2 + \omega^2)$$

which is the same as the characteristic function of

$$\Lambda_i = \begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix}$$

In other words, $\Lambda'_i$ is the diagonalized form of $\Lambda_i$ and exploiting the fact that we change the blocks in a block diagonal matrix at will, we can replace all complex $\Lambda'_i$ blocks with their corresponding real forms and corresponding diagonalization matrix is also updated accordingly. Please note that we leave the real eigenvalues the same as $1 \times 1$ block matrices.

6. (Optional) Left to the students.

7. (Optional) Left to the students.