The University of Texas at Austin
Department of Electrical and Computer Engineering

EE362K: Introduction to Automatic Control—Fall 2009

Problem Set Four

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This problem set should serve as a start for your review for the test. It covers the concepts introduced in Chapter 5. This includes homogeneous and particular solutions to LTI systems (CT and DT); The matrix exponential; Jordan Canonical Form and linear algebra; and stability of LTI systems. Also, while it also covers some of the earlier concepts discussed, it is not exhaustive. We had a lot of practice with linearization in class and in previous problem sets, so while that is an important concept, it is not emphasized in this one. It is definitely important for the midterm.

1. Consider the matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & 2 & 8 \\ 0 & 0 & 0 & -2 & -5 \\ 0 & 0 & 0 & 4 & -7 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

   (a) If you can, write down the Jordan Canonical Form (JCF) of this matrix and explain which results you are using in order to arrive to your answer. If you cannot, explain why there is ambiguity.

   (b) Is the system $\dot{x} = Ax$ stable, unstable, or stable i.s.L., for A the above matrix?

   (c) Is the system $x[k+1] = Ax[k]$ stable, unstable, or stable i.s.L., for A the above matrix?

2. Suppose A is a 7×7 symmetric matrix with characteristic polynomial:

$$p_A(\lambda) = 3 \cdot \lambda \cdot (\lambda + 2)^2 \cdot (\lambda + 1)^3 \cdot (\lambda + 12).$$

   (a) If you can, write down the Jordan Canonical Form (JCF) of this matrix and explain which results you are using in order to arrive to your answer. If you cannot, explain why there is ambiguity.

   (b) Is the system $\dot{x} = Ax$ stable, unstable, or stable i.s.L., for A the above matrix?

   (c) Is the system $x[k+1] = Ax[k]$ stable, unstable, or stable i.s.L., for A the above matrix?

3. Now suppose A is a 7×7 matrix with characteristic polynomial:

$$p_A(\lambda) = 3 \cdot \lambda^2 \cdot (\lambda + 1/2)^2 \cdot (\lambda + 1/3)^2 \cdot (\lambda + 1/5).$$

   (a) If you can, write down the Jordan Canonical Form (JCF) of this matrix and explain which results you are using in order to arrive to your answer. If you cannot, explain why there is ambiguity.

(b) Can you determine if the system $\dot{x} = Ax$ is stable, unstable, or stable i.s.L., for A the above matrix?

(c) Can you determine if the system $x[k+1] = Ax[k]$ is stable, unstable, or stable i.s.L., for A the above matrix?

4. Consider the matrix
$$A = \begin{bmatrix} -2 & \alpha \\ \alpha & -2 \end{bmatrix}$$

(a) For what values of α is the system $\dot{x} = Ax$ stable?

(b) For what values of α is the system $x[k+1] = Ax[k]$ stable?

5. Exercise 5.6 from the book.

6. Exercise 5.8 from the book.

7. Exercise 4.15, part (a).

8. In analogy to our denition of positive and semidenite functions, a matrix M is called positive denite if $x^\top Mx > 0$ for all nonzero vectors x. A matrix M is called positive semidenite if $x^\top Mx0$ for all vectors x.

- Fact 1: M is positive denite if and only if all its eigenvalues are positive.
- Fact 2: M is positive semidenite if and only if all its eigenvalues are nonnegative.

Consider a LTI CT system: $\dot{x} = Ax$. Consider a candidate Lyapunov function of the form $V(x) = x^\top Px$ for some symmetric $n \times n$ matrix P . Compute conditions on A and P for $V(x)$ to be a Lyapunov function for the system $\dot{x} = Ax$.

9. Now consider the same problem, but for discrete time. A Lyapunov function in discrete time must satisfy: $V(x)$ positivedefinite, and the discrete analog of energy dissipation: $V(x(k+1)) - V(x(k)) < 0$. Consider now the LTI DC system: $x[k+1] = Ax[k]$. Consider a candidate Lyapunov function of the form $V(x) = x^\top Px$ for some symmetric $n \times n$ matrix P . Compute conditions on A and P for $V(x)$ to be a Lyapunov function.

10. Exercise 5.5 from the book. You also need to assume that Q is positive definite.

11. Recall that if a positive definite function $V(x)$ fails to be a Lyapunov function for a particular dynamical system, then that system may nevertheless be stable. Explain this to yourselves, and look over one of those examples in the book (we also did one in class).

12. Show that the range of a matrix,
$$V = \{y \,|\, y = Ax, \quad x \in \mathbb{R}^n\},$$
is a vector space. Also show that V is the span of the columns of A.

13. Suppose V_1 and V_2 are both vector subspaces of $\mathbb{R}^n$. Show that $V_1 \cap V_2$ is a vector space.

14. Suppose V_1 and V_2 are both vector subspaces of $\mathbb{R}^n$. Under what conditions is $V_1 \cup V_2$ a vector space? Give an example of a V_1 and V_2 where their union is a vector space, and an example where their union is not a vector space.

15. Suppose $\boldsymbol{A}$ is an $n \times n$ matrix, with eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, and eigenvalues $\lambda_1, \ldots, \lambda_n$. Compute the eigenvalues and eigenvectors of:

- $\boldsymbol{A}^k$.
- $e^{\boldsymbol{A}t}$.
- TAT^{-1}, for T an invertible matrix.

16. Reachability: consider the discrete-time system: $x[k+1] = Ax[k] + Bu[k]$.

(a) Is the following system reachable?

$$A = \begin{bmatrix} -1 & 4 & 3 \\ 2 & 6 & -1 \\ -2 & -5 & 2 \end{bmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(b) What about this one?

$$A = \begin{bmatrix} -1 & 4 & 3 \\ 2 & 6 & -1 \\ -2 & -5 & 2 \end{bmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(c) Can you find a $n \times 1$ matrix B so that the following system is reachable?

$$A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

17. Consider the discrete time LTI system

$$\boldsymbol{x}[k+1] = A\boldsymbol{x}[k] + Bu[k],$$

where we have:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Is this system reachable?

(b) Suppose $x(0) = \mathbf{0}$. Compute a control that takes the initial state to the final state $x(T) = (1, 0, 0)$, in the least time possible.