The University of Texas at Austin
Department of Electrical and Computer Engineering

EE362K: Introduction to Automatic Control—Fall 2009

Problem Set Four Solutions

C. Caramanis — Not Due.

This problem set should serve as a start for your review for the test. It covers the concepts introduced in Chapter 5. This includes homogeneous and particular solutions to LTI systems (CT and DT); The matrix exponential; Jordan Canonical Form and linear algebra; and stability of LTI systems. Also, while it also covers some of the earlier concepts discussed, it is not exhaustive. We had a lot of practice with linearization in class and in previous problem sets, so while that is an important concept, it is not emphasized in this one. It is definitely important for the midterm.

1. Consider the matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & 2 & 8 \\ 0 & 0 & 0 & -2 & -5 \\ 0 & 0 & 0 & 4 & -7 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

   (a) If you can, write down the Jordan Canonical Form (JCF) of this matrix and explain which results you are using in order to arrive to your answer. If you cannot, explain why there is ambiguity.

   (b) Is the system $\dot{x} = Ax$ stable, unstable, or stable i.s.L., for A the above matrix?

   (c) Is the system $x[k+1] = Ax[k]$ stable, unstable, or stable i.s.L., for A the above matrix?

   Solution:

   (a) The matrix is upper triangular, and hence the eigenvalues are precisely the values on the diagonal. Thus the eigenvalues are: $\lambda = 1, -3, 0, 4, 6$. Since these values are distinct, the JCF of the matrix must be composed of 1×1 blocks, i.e., it is diagonal, with the above values along the diagonal.

   (b) The continuous time system is unstable, since there is an eigenvalue (several, in fact) with strictly positive real part.

   (c) The discrete time system is also unstable, since there is an eigenvalue (several, in fact) strictly outside the unit circle.

2. Suppose A is a 7×7 symmetric matrix with characteristic polynomial:

$$p_A(\lambda) = 3 \cdot \lambda \cdot (\lambda + 2)^2 \cdot (\lambda + 1)^3 \cdot (\lambda + 12).$$

   (a) If you can, write down the Jordan Canonical Form (JCF) of this matrix and explain which results you are using in order to arrive to your answer. If you cannot, explain why there is ambiguity.

(b) Is the system $\dot{x} = Ax$ stable, unstable, or stable i.s.L., for A the above matrix?

(c) Is the system $x[k+1] = Ax[k]$ stable, unstable, or stable i.s.L., for A the above matrix?

Solution:

(a) The eigenvalues are not distinct. They are: $\lambda = 0, -2, -2, -1, -1, -1, -12$. However, since the matrix is symmetric, we know it is diagonalizable, and hence its JCF must be the diagonal matrix with the values above in its diagonal.

(b) The continuous time system is neutrally aka marginally stable. This is because there are no eigenvalues strictly in the right half plane, and the one with real part equal to zero is in a 1×1 Jordan block.

(c) The discrete time system is unstable, since there is an eigenvalue (several, in fact) strictly outside the unit circle.

3. Now suppose A is a 7×7 matrix with characteristic polynomial:

$$p_A(\lambda) = 3 \cdot \lambda^2 \cdot (\lambda + 1/2)^2 \cdot (\lambda + 1/3)^2 \cdot (\lambda + 1/5).$$

(a) If you can, write down the Jordan Canonical Form (JCF) of this matrix and explain which results you are using in order to arrive to your answer. If you cannot, explain why there is ambiguity.

(b) Can you determine if the system $\dot{x} = Ax$ is stable, unstable, or stable i.s.L., for A the above matrix?

(c) Can you determine if the system $x[k+1] = Ax[k]$ is stable, unstable, or stable i.s.L., for A the above matrix?

Solution:

(a) In this case the eigenvalues are: $\lambda = 0, 0, -1/2, -1/2, -1/3, -1/3, -1/5$. We cannot write down the JCF from only this information. Indeed, there are 8 candidate JCFs for a matrix with these eigenvalues:

$$A = \begin{bmatrix} 0 & ? & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & ? & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & ? & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} \end{bmatrix}$$

The three '?' determine if the eigenvalues 0, $-1/2$, and $-1/3$ are in a single 2×2 block, or if they are in 2 1×1 blocks. There are $2 \times 2 \times 2 = 8$ possible combinations.

(b) We cannot determine if the continuous time system is unstable or marginally stable. If the two zeros are in the same Jordan block, then the continuous time system is unstable. If they are in separate blocks, then the continuous time system is marginally stable (aka neutrally stable, aka stable i.s.L.).

(c) Even though we cannot determine the JCF or stability in the continuous time setting, we *can* determine stability in the discrete time setting: the discrete time system is stable, since all the eigenvalues are strictly inside the unit circle, and hence the size of the Jordan blocks is irrelevant (from a stability perspective).

4. Consider the matrix

$$A = \begin{bmatrix} -2 & \alpha \\ \alpha & -2 \end{bmatrix}$$

   (a) For what values of α is the system $\dot{x} = Ax$ stable?
   (b) For what values of α is the system $x[k+1] = Ax[k]$ stable?

   Solution:

   The characteristic polynomial of A can be found to be $p_A(\lambda) = (\lambda+2)^2 - \alpha^2$. Solving for $p_A(\lambda) = 0$ gives the eigenvalues $\lambda_1 = -2+\alpha$, $\lambda_2 = -2-\alpha$. We assume $\alpha \in \mathbb{R}$.

   (a) For $\dot{x} = Ax$ to be asymptotically stable, we require $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$. This means $-2+\alpha < 0$ and $-2-\alpha < 0$ which gives $-2 < \alpha < 2$. Also, as A is symmetric, it is diagonalizable and the Jordan blocks for λ_1, λ_2 are of size 1×1. This means for $\alpha = \pm 2$, which produce the eigenvalue 0, the system is stable i.s.L.
   (b) For $x[k+1] = Ax[k]$ to be stable (asymptotic or i.s.L.), we require $|\lambda_1| \leq 1$ and $|\lambda_2| \leq 1$. This means $|-2+\alpha| \leq 1$ and $|-2-\alpha| \leq 1$ which give $\alpha \in [1,3]$ and $\alpha \in [-3,-1]$ respectively. But $[1,3] \cap [-3,-1] = \phi$. This means the system is unstable $\forall\, \alpha \in \mathbb{R}$.

5. Exercise 5.6 from the book.

   Solution:

   (a) There are several ways to choose A, B, C and $u(t)$. One way is

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, u(t) = \cos t$$

   Assuming initial condition $x(0) = 0$ gives

$$y(t) = C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = -\frac{1}{2}te^{-t} + \frac{1}{2}\sin t \quad \text{(verify yourself)}$$

   which is not periodic.

   (b) Suppose the system has eigenvalue $\lambda = 0$ with Jordan block J $(p \times p)$.

$$J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \Rightarrow e^{Jt} = \begin{bmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{p-1}}{(p-1)!} \\ 0 & 1 & t & & \frac{t^{p-2}}{(p-2)!} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & 1 & t \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

   Since J is non-trivial, we have $p > 1$ which means at-least the element 't' is present in e^{Jt}. This means the magnitude of at-least one element of e^{Jt} becomes arbitrarily large as $t \to \infty$, which makes the system unstable.

   Next, suppose the system has eigenvalue $\lambda = i\omega$ $(\omega \in \mathbb{R})$, with Jordan block J $(p \times p)$.

$$J = \begin{bmatrix} i\omega & 1 & 0 & \cdots & 0 \\ 0 & i\omega & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & i\omega & 1 \\ 0 & 0 & \cdots & 0 & i\omega \end{bmatrix} \Rightarrow e^{Jt} = \begin{bmatrix} e^{i\omega t} & te^{i\omega t} & \frac{t^2}{2}e^{i\omega t} & \cdots & \frac{t^{p-1}}{(p-1)!}e^{i\omega t} \\ 0 & e^{i\omega t} & te^{i\omega t} & & \frac{t^{p-2}}{(p-2)!}e^{i\omega t} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & e^{i\omega t} & te^{i\omega t} \\ 0 & 0 & \cdots & 0 & e^{i\omega t} \end{bmatrix}$$

Since J is non-trivial, we have $p > 1$ which means at-least the element '$te^{i\omega t}$' is present in e^{Jt}. As $|te^{i\omega t}| = |t|$, this means the magnitude of at-least one element of e^{Jt} becomes arbitrarily large as $t \to \infty$, which makes the system unstable.

6. Exercise 5.8 from the book.

7. Exercise 4.15, part (a).

   Solution:

8. The equilibrium condition is given by

$$J_p \omega_0^2 \sin\theta \cos\theta + m_p g l \sin\theta = 0 \Rightarrow (J_p \omega_0^2 \cos\theta + m_p g l) \sin\theta = 0$$

This always has the solutions $\theta = n\pi$, where even n corresponds to the pendulum standing upright and odd n corresponds to the pendulum hanging down. If $J_p \omega_0^2 > m_p g l$, there are other solutions given by $\theta_0 = \pm \arccos\left(-m_p g l / J_p \omega_0^2\right)$.

To find the stability of the equilibria we consider the state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$. Then the system can be written as

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \underbrace{\begin{pmatrix} x_2 \\ \frac{m_p g l}{J_p} \sin x_1 + \frac{\omega_0^2}{2} \sin 2x_1 \end{pmatrix}}_{f(x_1, x_2)}$$

Computing the Jacobian of $f(x_1, x_2)$ gives

$$J(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ \frac{m_p g l}{J_p} \cos x_1 + \omega_0^2 \cos 2x_1 & 0 \end{pmatrix}$$

Evaluating $J(x_1, x_2)$ at the equilibria when the pendulum is hanging up ($(x_1, x_2) = (0, 0)$) and down ($(x_1, x_2) = (\pi, 0)$) respectively, we get

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ \frac{m_p g l}{J_p} + \omega_0^2 & 0 \end{pmatrix}, \quad J(\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{m_p g l}{J_p} + \omega_0^2 & 0 \end{pmatrix}$$

The equilibrium $(0, 0)$ is unstable since $m_p g l / J_p + \omega_0^2 > 0$, the equilibrium $(\pi, 0)$ is unstable if $\omega_0 \geq \sqrt{m_p g l / J_p}$ and stable (but not asymptotically stable) if $\omega_0 < \sqrt{m_p g l / J_p}$. For the equilibria when the pendulum is at the angle $\pm\theta_0$ ($(x_1, x_2) = (\theta_0, 0)$), we have

$$J(\theta_0, 0) = \begin{pmatrix} 0 & 1 \\ \frac{1}{\omega_0^2}\left(\frac{m_p g l}{J_p}\right)^2 - \omega_0^2 & 0 \end{pmatrix}$$

The equilibria exist only if $\omega_0 > \sqrt{m_p g l / J_p}$ and they are then stable.

9. In analogy to our denition of positive and semidenite functions, a matrix M is called positive denite if $x^\top M x > 0$ for all nonzero vectors x. A matrix M is called positive semidenite if $x^\top M x 0$ for all vectors x.

   - Fact 1: M is positive denite if and only if all its eigenvalues are positive.
   - Fact 2: M is positive semidenite if and only if all its eigenvalues are nonnegative.

Consider a LTI CT system: $\dot{x} = Ax$. Consider a candidate Lyapunov function of the form $V(x) = x^\top Px$ for some symmetric $n \times n$ matrix P . Compute conditions on A and P for $V(x)$ to be a Lyapunov function for the system $\dot{x} = Ax$.

Solution: $V(x) = x^T Px$ is a Lyapunov function for the system $\dot{x} = Ax$ if

- $V(x)$ is positive definite i.e. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$:
  $V(0) = (0)^T P(0) = 0$. $V(x) = x^T Px > 0$ for $x \neq 0$ only if P is positive definite.
- $\dot{V}(x)$ is negative semi-definite i.e. $\dot{V}(0) = 0$ and $\dot{V}(x) \leq 0$ for $x \neq 0$:
  $\dot{V}(x)$ can be computed as

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T Px + x^T P\dot{x} \\ &= (Ax)^T Px + x^T P(Ax) \\ &= x^T(A^T P + PA)x \end{aligned}$$

  $\dot{V}(0) = (0)^T(A^T P + PA)(0) = 0$. $\dot{V}(x) = x^T(A^T P + PA)x \leq 0$ for $x \neq 0$ only if $A^T P + PA$ is negative semi-definite.

Hence, $V(x) = x^T Px$ is a Lyapunov function for the system if P is a positive definite matrix and $A^T P + PA$ is negative semi-definite matrix.

10. Now consider the same problem, but for discrete time. A Lyapunov function in discrete time must satisfy: $V(x)$ positivedefinite, and the discrete analog of energy dissipation: $V(x(k+1)) - V(x(k)) < 0$. Consider now the LTI DC system: $x[k+1] = Ax[k]$. Consider a candidate Lyapunov function of the form $V(x) = x^\top Px$ for some symmetric $n \times n$ matrix P . Compute conditions on A and P for $V(x)$ to be a Lyapunov function.

    Solution: $V(x) = x^T Px$ is a Lyapunov function for the system $x[k+1] = Ax[k]$ if

    - $V(x)$ is positive definite i.e. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$:
      $V(0) = (0)^T P(0) = 0$. $V(x) = x^T Px > 0$ for $x \neq 0$ only if P is positive definite.
    - $V(x[k+1]) - V(x[k]) < 0$ for $x[k] \neq 0$:
      $V(x[k+1]) - V(x[k])$ can be computed as

$$\begin{aligned} V(x[k+1]) - V(x[k]) &= (x[k+1])^T Px[k+1] - (x[k])^T Px[k] \\ &= (Ax[k])^T P(Ax[k]) - (x[k])^T Px[k] \\ &= (x[k])^T(A^T PA - P)x[k] \end{aligned}$$

      $V(x[k+1]) - V(x[k]) = (x[k])^T(A^T PA - P)x[k] < 0$ for $x[k] \neq 0$ only if $A^T PA - P$ is negative definite.

    Hence, $V(x) = x^T Px$ is a Lyapunov function for the system if P is a positive definite matrix and $A^T PA - P$ is a negative definite matrix.

11. Exercise 5.5 from the book. You also need to assume that Q is positive definite.

    Solution:

    P is positive definite as

$$\begin{aligned} x^T Px &= \int_0^\infty x^T e^{A^T\tau} Q e^{A\tau} x d\tau \\ &= \int_0^\infty (e^{A\tau}x)^T Q(e^{A\tau}x) d\tau > 0 \qquad \text{(positive definiteness of } Q) \end{aligned}$$

Also, $A^TP + PA$ is negative definite as

$$\begin{aligned} A^TP + PA &= \int_0^\infty (A^T e^{A^T\tau} Q e^{A\tau} + e^{A^T\tau} Q e^{A\tau} A) d\tau \\ &= \int_0^\infty \frac{d}{d\tau}(e^{A^T\tau} Q e^{A\tau}) d\tau \qquad \text{(using chain rule)} \\ &= e^{A^T\tau} Q e^{A\tau}|_0^\infty \end{aligned}$$

From the fact that $Re(\lambda_j) < 0$ for A, it follows that $e^{At} \to 0$ as $t \to \infty$ i.e.

$$A^TP + PA = e^{A^T\tau} Q e^{A\tau}|_0^\infty = -Q$$

which is negative definite. Since P is positive definite and $A^TP + PA$ is negative definite, we have that $V(x) = x^TPx$ is a Lyapunov function.

12. Recall that if a positive definite function $V(x)$ fails to be a Lyapunov function for a particular dynamical system, then that system may nevertheless be stable. Explain this to yourselves, and look over one of those examples in the book (we also did one in class).

13. Show that the range of a matrix,

$$V = \{y \,|\, y = Ax, \quad x \in \mathbb{R}^n\},$$

is a vector space. Also show that V is the span of the columns of A.

Solution: $V = \{y|y = Ax, x \in \mathbb{R}^n\}$ is a vector space as

$$\begin{aligned} y_1, y_2 \in V &\Rightarrow y_1 = Ax_1, y_2 = Ax_2 \text{ for some } x_1, x_2 \in \mathbb{R}^n \\ &\Rightarrow \alpha y_1 + \beta y_2 = A(\alpha x_1 + \beta x_2) \text{ for } \alpha, \beta \in \mathbb{R}^n \\ &\Rightarrow \alpha y_1 + \beta y_2 \in V \end{aligned}$$

If $A = [A_1 \; A_2 \ldots A_n]$ ($A_i \in \mathbb{R}^n$ denotes the ith column of A) and $x = [x_1 \; x_2 \ldots x_n]^T$ ($x_i \in \mathbb{R}$) then $Ax = x_1A_1 + x_2A_2 + \ldots + x_nA_n$. This gives

$$\begin{aligned} V &= \{y|y = Ax, x \in \mathbb{R}^n\} \\ &= \{y|y = x_1A_1 + x_2A_2 + \ldots + x_nA_n, x_1, x_2 \ldots, x_n \in \mathbb{R}\} \\ &= \{x_1A_1 + x_2A_2 + \ldots + x_nA_n | x_1, x_2 \ldots, x_n \in \mathbb{R}\} \\ &= \text{span}(A_1, A_2, \ldots, A_n) \end{aligned}$$

i.e. V is the span of columns of A.

14. Suppose V_1 and V_2 are both vector subspaces of $\mathbb{R}^n$. Show that $V_1 \cap V_2$ is a vector space.

Solution:

Since V_1 and V_2 are vector spaces of $\mathbb{R}^n$, both of them have the origin 0 as an element. This means $V_1 \cap V_2 \neq \phi$. If any one of V_1 or V_2 is $\{0\}$, then $V_1 \cap V_2 = \{0\}$, which is a trivial vector space. Hence, we assume $V_1 \neq \{0\}, V_2 \neq \{0\}$ so that $V_1 \cap V_2$ has more than one element. Then $V_1 \cap V_2$ is a vector space as

$$\begin{aligned} x, y \in V_1 \cap V_2 &\Rightarrow x, y \in V_1 \text{ and } x, y \in V_2 \\ &\Rightarrow \alpha x + \beta y \in V_1 \text{ and } \alpha x + \beta y \in V_2 \text{ for } \alpha, \beta \in \mathbb{R}^n \quad (*) \\ &\Rightarrow \alpha x + \beta y \in V_1 \cap V_2 \end{aligned}$$

$(*)$ is due to the fact that V_1 and V_2 are vector spaces

15. Suppose V_1 and V_2 are both vector subspaces of $\mathbb{R}^n$. Under what conditions is $V_1 \cup V_2$ a vector space? Give an example of a V_1 and V_2 where their union is a vector space, and an example where their union is not a vector space.

    Solution: We claim that for $V_1 \cup V_2$ to be a vector space, where V_1 and V_2 are vector spaces of $\mathbb{R}^n$, either $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$. Clearly, if $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$ then $V_1 \cup V_2 = V_2$ or V_1, which is a vector space. Now, suppose there exists vector spaces V_1 and V_2 such that neither is a subset of the other and yet $V_1 \cup V_2$ is a vector space. This means $V_1 \neq \{0\}, V_2 \neq \{0\}$, where 0 is the origin, as $\{0\}$ is a subset of any vector space of $\mathbb{R}^n$. This gives $V_1 \backslash V_2 \neq \phi$ and $V_2 \backslash V_1 \neq \phi$. Let $x \in V_1 \backslash V_2$ and $y \in V_2 \backslash V_1$. Since $V_1 \cup V_2$ is a subspace, $z = x + y \in V_1 \cup V_2$. So, $z \in V_1$ or $z \in V_2$. If $z \in V_1$ then $z - x \in V_1$ or $y \in V_1$, which is a contradiction. Similarly if $z \in V_2$, we get a contradiction. Thus, one of V_1 or V_2 is a subset of the other.

    An example of V_1 and V_2, whose union is a vector space, is $V_1 = \{0\}$ and $V_2 = \mathbb{R}^n$ ($V_1 \cup V_2 = \mathbb{R}^n$). An example of V_1 and V_2, whose union is not a vector space, is

$$\begin{aligned} V_1 &= \{(\alpha, 0, \ldots, 0, 0) | \alpha \in \mathbb{R}\} \\ V_2 &= \{(0, \beta, \ldots, 0, 0) | \beta \in \mathbb{R}\} \end{aligned}$$

16. Suppose $\boldsymbol{A}$ is an $n \times n$ matrix, with eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, and eigenvalues $\lambda_1, \ldots, \lambda_n$. Compute the eigenvalues and eigenvectors of:

    - $\boldsymbol{A}^k$.
    - $e^{\boldsymbol{A}t}$.
    - TAT^{-1}, for T an invertible matrix.

    Solution:

    - $Av_i = \lambda_i v_i \Rightarrow$

$$A^k v_i = A^{k-1}(Av_i) = A^{k-1}(\lambda_i v_i) = \lambda_i (A^{k-1} v_i) = \ldots = \lambda_i^2 (A^{k-2} v_i) = \ldots = \lambda_i^k v_i$$

      This means the eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ and the corresponding eigenvectors are $v_1, v_2, \ldots, v_n$ respectively.

    - $Av_i = \lambda_i v_i$ and $A^k v_i = \lambda_i^k v_i \Rightarrow$

$$e^{At} v_i = \left(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \right) v_i = \sum_{k=0}^{\infty} \frac{(A^k v_i) t^k}{k!} = \sum_{k=0}^{\infty} \frac{(\lambda_i^k v_i) t^k}{k!} = \left(\sum_{k=0}^{\infty} \frac{\lambda_i^k t^k}{k!} \right) v_i = e^{\lambda_i t} v_i$$

      This means the eigenvalues of e^{At} are $e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}$ and the corresponding eigenvectors are $v_1, v_2, \ldots, v_n$ respectively.

    - $Av_i = \lambda_i v_i \Rightarrow$

$$AT^{-1} T v_i = \lambda_i v_i \Rightarrow TAT^{-1} T v_i = T \lambda_i v_i \Rightarrow (TAT^{-1})(Tv_i) = \lambda_i (Tv_i)$$

      This means the eigenvalues of TAT^{-1} are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the corresponding eigenvectors are $Tv_1, Tv_2, \ldots, Tv_n$ respectively.

17. Reachability: consider the discrete-time system: $x[k+1] = Ax[k] + Bu[k]$.

(a) Is the following system reachable?

$$A = \begin{bmatrix} -1 & 4 & 3 \\ 2 & 6 & -1 \\ -2 & -5 & 2 \end{bmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Solution: Yes. Check that the rank of the matrix

$$W_r = [\ B \quad AB \quad A^2B\ ],$$

is indeed 3.

(b) What about this one?

$$A = \begin{bmatrix} -1 & 4 & 3 \\ 2 & 6 & -1 \\ -2 & -5 & 2 \end{bmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Solution No. Check that the rank of the matrix

$$W_r = [\ B \quad AB \quad A^2B\ ],$$

is 2.

(c) Can you find a $n \times 1$ matrix B so that the following system is reachable?

$$A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Solution: No. For any B, we have:

$$W_r = [\ B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B\ ] = [\ B \quad B \quad \cdots \quad B\ ],$$

which cannot have rank more than 1.

18. Consider the discrete time LTI system

$$\boldsymbol{x}[k+1] = A\boldsymbol{x}[k] + Bu[k],$$

where we have:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Is this system reachable?

(b) Suppose $x(0) = \mathbf{0}$. Compute a control that takes the initial state to the final state $x(T) = (1, 0, 0)$, in the least time possible.

Solution:

(a) Yes, the system is reachable.

$$W_r = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 11 \end{bmatrix},$$

which has rank 3.

(b) We know that since the system is reachable, we can reach any state in three steps. Can we do it in one or two? The answer is found to be no, by considering the set of states reachable by time 1, and then by time 2.

The states reachable from the origin in one time step is $\text{span}\{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^\top\}$. The states reachable from the origin in two time steps is $\text{span}\{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^\top, \begin{pmatrix} 0 & 1 & -4 \end{pmatrix}^\top\}$. It is plain to see that the vector $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^\top$ is not in either of these sets.