The University of Texas at Austin
Department of Electrical and Computer Engineering

EE362K: Introduction to Automatic Control—Fall 2009

PROBLEM SET FIVE SOLUTION

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This problem set focuses on the new concepts introduced in the last two classes: reachability and state feedback. As usual, we also work in some exercise with important concepts from linear algebra.

1. Reachable Canonical Form: Consider the system from above:

$$A = \begin{bmatrix} -1 & 4 & 3 \\ 2 & 6 & -1 \\ -2 & -5 & 2 \end{bmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   (a) What is the reachable canonical form, $\tilde{A}$, for the matrix A?

   (b) In class we showed that $\tilde{A}$ and A are related by the relationship: $\tilde{A} = TAT^{-1}$, for some invertible matrix T.[1] We showed in class that this kind of transformation preserves eigenvalues, and therefore also determinants. Use Matlab (you can compute by hand, if you wish) to verify that $\tilde{A}$ and A have the same eigenvalues, and the same determinant (equal to the product of the eigenvalues).

   (c) Follow the procedure we outlined in class (and also outlined in the text) to compute the invertible matrix T that can be used to transform from A to $\tilde{A}$. Check your answer to see that this indeed has worked.

Solution:

   (a) The characteristic polynomial of A is given by

$$p_A(\lambda) = |\lambda I - A| = \lambda^3 - 7\lambda^2 - 3\lambda + 9$$

   This gives the reachable canonical form

$$\tilde{A} = \begin{bmatrix} 7 & 3 & -9 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; \tilde{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

   (b) The characteristic polynomial of $\tilde{A}$ is given by

$$p_{\tilde{A}}(\lambda) = |\lambda I - \tilde{A}| = \lambda^3 - 7\lambda^2 - 3\lambda + 9$$

   Since $p_{\tilde{A}}(\lambda) = p_A(\lambda)$, A and $\tilde{A}$ have the same set of eigenvalues. Their determinant values are also same as determinant of a matrix is equal to the product of its eigenvalues.

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[1] This is known as a *similarity transformation*.

(c) According to Equation (6.8) of the textbook, and also as we derived in class, $T = \tilde{W}_r W_r^{-1}$, where $\tilde{W}_r$ is the reachability matrix for $(\tilde{A}, \tilde{B})$. This gives

$$W_r = \begin{bmatrix} 0 & 3 & -1 \\ 0 & -1 & -2 \\ 1 & 2 & 3 \end{bmatrix}, \ \tilde{W}_r = \begin{bmatrix} 1 & 7 & 52 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow T = \tilde{W}_r W_r^{-1} = \begin{bmatrix} -\frac{39}{7} & -\frac{152}{7} & 1 \\ -\frac{5}{7} & -\frac{22}{7} & 0 \\ -\frac{1}{7} & -\frac{3}{7} & 0 \end{bmatrix}$$

T can also be found by hand by solving $\tilde{B} = TB$ and $\tilde{A}T = TA$ simultaneously.

2. Show that the change of coordinates we are so freely using is not bogus: Consider the system:

$$\dot{x} = Ax + Bu.$$

Let T be some invertible matrix, and consider the change of coordinates $z = Tx$.

- Write down the dynamics for the system in the z coordinates.
- Show that if the original system is reachable, then so is the new one, and then conversely, if the new system is reachable, then the original one must be reachable. Therefore, you are showing that reachability is a physical property of the system and is not representation-dependent.

Solution:

(a) Since T is invertible, $x = T^{-1}z$ and $\dot{x} = T^{-1}\dot{z}$ which gives:

$$\begin{aligned} \dot{x} = Ax + Bu \ &\Rightarrow \ T^{-1}\dot{z} = AT^{-1}z + Bu \\ &\Rightarrow \ \dot{z} = TAT^{-1}z + TBu \\ &\Rightarrow \ \dot{z} = \hat{A}z + \hat{B}u \end{aligned}$$

where $\hat{A} = TAT^{-1}$ and $\hat{B} = TB$.

(b) It can be verified that $\hat{A}^k = TAT^{-1}TAT^{-1} \dots TAT^{-1} = TA^kT^{-1}$. Further, $\hat{A}^k\hat{B} = TA^kT^{-1}TB = TA^kB$. The reachability matrix of $(\hat{A}, \hat{B})$ is then given by

$$\begin{aligned} \hat{W}_r &= [\ \hat{B} \ \ \hat{A}\hat{B} \ \ \dots \ \ \hat{A}^{n-1}\hat{B}\ ] \\ &= [\ TB \ \ TAB \ \ \dots \ \ TA^{n-1}B\ ] \\ &= TW_r \end{aligned}$$

where W_r is the reachability matrix of (A, B). Since T is invertible, $rank(TW_r) = rank(W_r)$ which means $rank(\hat{W}_r) = rank(W_r)$. So, the system is reachable in z coordinate if the system is reachable in x coordinate.

3. Eigenvalue Assignment: Consider the system:

$$A = \begin{bmatrix} -1 & 5 & 2 \\ -2 & 4 & -1 \\ -2 & 3 & 2 \end{bmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(a) What is the reachable canonical form, $\tilde{A}$, for the matrix A?

(b) Compute the invertible matrix T that can be used to transform from A to $\tilde{A}$ (as you did in the problem above, and as we did in class a couple of times).

(c) Is the open-loop system stable?

(d) Design state feedback $u = -Kx$ so that the closed loop system is stable, with closed-loop eigenvalue $\lambda_{cl} = \{-1, -2, -3\}$.

Solution:

(a) In order to find the reachable canonical form, $(\tilde{A}, \tilde{B})$, we can construct the characteristic polynomial of matrix A and then use 6.6 from the book to order them in the desired format. The characteristic function of the given A matrix is given by

$$\Phi_A(\lambda) = |\lambda I - A| = \lambda^3 - 5\lambda^2 + 19\lambda - 23$$

and hence,

$$\tilde{A} = \begin{bmatrix} 5 & -19 & 23 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(b) As suggested by the textbook in equation 6.8, we can directly obtain T using the reachability matrices for the original and canonical forms:

$$W_r = \begin{bmatrix} 0 & 3 & -1 \\ 0 & -1 & -2 \\ 1 & 2 & 3 \end{bmatrix}, \quad \tilde{W}_r = \begin{bmatrix} 1 & 7 & 52 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \quad T = \tilde{W}_r W_r^{-1} = \begin{bmatrix} \frac{21}{23} & -\frac{27}{23} & 1 \\ \frac{5}{23} & -\frac{13}{23} & 0 \\ -\frac{1}{23} & -\frac{2}{23} & 0 \end{bmatrix}.$$

(c) Solving $\Phi_A(\lambda) = 0$ we can find the eigenvalues of the open loop system as:

$$\Phi_A(\lambda) = 0 \quad \Rightarrow \quad \lambda = \{1.7222, 1.6389 \pm 3.2663i\}$$

Since $Re(\lambda) > 0$, the open loop system is unstable.

(d) As discussed in the class, we can place the eigenvalues of the system anywhere we like using a linear state feedback, $u = -Kx$. You can also go through equations 6.15 to 6.21 to catch the idea. According to the given closed loop eigenvalues (λ_{cl}), the desired closed loop characteristic polynomial would be:

$$\Phi_{A,cl}(\lambda) = \prod_i (\lambda - \lambda_{cl,i}) = (\lambda + 1)(\lambda + 2)(\lambda + 3) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

according to 6.17, we know that the coefficients of this polynomial are in the form of $a_i + \tilde{k}_i$ where a_is are the coefficients of the open loop characteristic polynomial and $\tilde{k}_i$s are the elements of $\tilde{K}$, where $\tilde{K}$ is the feedback coefficients matrix for the system represented in the rechability canonical form. Hence, $\tilde{K}$ would be given by:

$$\tilde{K} = \begin{bmatrix} 6 - (-5) & 11 - 19 & 6 - (-23) \end{bmatrix} = \begin{bmatrix} 11 & -8 & 29 \end{bmatrix}$$

and K is obtained by transforming the system back to its original coordinates as:

$$K = \tilde{K}T = \begin{bmatrix} 11 & -8 & 29 \end{bmatrix} \begin{bmatrix} \frac{21}{23} & -\frac{27}{23} & 1 \\ \frac{5}{23} & -\frac{13}{23} & 0 \\ -\frac{1}{23} & -\frac{2}{23} & 0 \end{bmatrix} = \begin{bmatrix} \frac{162}{23} & -\frac{251}{23} & 11 \end{bmatrix}$$

4. Robustness: In class I claimed that if we perturb a matrix by a small perturbation, then the location of the eigenvalues of the matrix will not change much. In particular, if the eigenvalues of a matrix are deep in the OLHP, then under a small perturbation, they will continue to be (somewhere in) the OLHP.

   (a) Using Matlab, compute a random matrix perturbation as follows: let $r = \text{randn}(3)$, and then let $R = (r + r')/2$. The first command generates a 3×3 matrix whose entries are generated in an independent and identically distributed (iid) fashion from a standard Gaussian distribution. The second command generates a symmetric matrix.

   (b) Using the same matrix A from above, and the same closed loop control $u = -Kx$ as above, consider the closed loop system matrix: $A_{cl} = ((A + R/p) - BK)$. Here, p is a normalization parameter. Find the closed-loop eigenvalues of the perturbed system for different values of p, i.e., $p = 10,000,\ 1,000,\ 100,\ 10,\ 5,\ 3,\ 2,\ 1$. When does the closed loop system become unstable?

   (c) Now choose a new feedback K so that the eigenvalues of the nominal (i.e., the unperturbed) closed loop matrix are at $-20, -20, -20$. Now when does the system become unstable? Use the same r as above, i.e., do not generate a new random matrix R.[2]

Solution:

(a) This is the sequence of the MATLAB commands we should run. The results might be different (actually they will almost surely, unless we have the same seed for the random number generator!) from yours but it does not matter.

```
>> r=randn(3)

r =

    0.1746   -0.5883    0.1139
   -0.1867    2.1832    1.0668
    0.7258   -0.1364    0.0593

>> R=(r+r')/2

R =

    0.1746   -0.3875    0.4199
   -0.3875    2.1832    0.4652
    0.4199    0.4652    0.0593
```

(b) Finding the closed roots of the characteristic function in matlab is quite easy so do not be scared. For given p, since we have A, R, B and K we can calculate A_{cl}. In MATLAB, the `poly()` function gives the characteristic polynomial of a matrix and the `roots()` function calculates its roots, combining these functions we can calculate the eigenvalues for each case in a single shot as `lambda=roots(poly(A+R/p-B*K))`. Here is a sample for $p = 10$, I have put all the setup in advance just to show you what are the values of A, R, B, K and p:

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[2] Note that the actual results will depend on the random matrix generated. Most likely a random perturbation will give you some eigenvalues in the RHP, but if not, just try generating a new matrix.

```
>> A

A =

    -1     5     2
    -2     4    -1
    -2     3     2

>> B

B =

     0
     0
     1

>> R

R =

    0.1746   -0.3875    0.4199
   -0.3875    2.1832    0.4652
    0.4199    0.4652    0.0593

>> K

K =

    7.0435  -10.9130   11.0000

>> p=10

p =

    10

>> lambda=roots(poly(A+R/p-B*K))

lambda =

  -3.8622
  -0.9481 + 0.5800i
  -0.9481 - 0.5800i
```

The results for the given values of p are tabulated in table 1.

As it can be seen the eigenvalues of the system go unstable for $p = 1$ and $p = 2$ as the role of perturbation in A becomes more significant.

p	λ_1	λ_2	λ_3
10,000	-3.0020	-1.9973	-1.0005
1,000	-3.0193	-1.9733	-1.0051
100	-3.1629	-1.7519	-1.0611
10	-3.8622	-0.9481 - 0.58i	-0.9481 + 0.58i
5	-4.2777	-0.6195 - 0.6579i	-0.6195 + 0.6579i
3	-4.6599	-0.2672 - 0.5457i	-0.2672 + 0.5457i
2	-5.0098	0.4706	-0.2523
1	-5.6941	2.6459	-0.5347

Table 1: Closed loop eigenvalues vs. p

(c) If we design the system for $\lambda_{cl} = -20, 20, 20$ the resulting K would be:

$$K = \begin{bmatrix} -\frac{753}{23} & -\frac{33154}{23} & 65 \end{bmatrix}$$

In order to find out when the system becomes unstable we keep decreasing p towards zero and find when we have at least one eigenvalue in RHP. For our R matrix we have: As it can be seen, for $p = 0.8$ and below, we have eigenvalues in the RHP. It should be noted that due to the fact that for this case we had the designed closed loop eigenvalues more deeply into the OLHP, we should have more perturbation to be thrown to the RHP and hence we should have less value for p to become unstable. In other words, the more deeply the closed loop eigenvalues are into the OLHP, the more the system is robust to perturbations in the A matrix.

p	λ_1	λ_2	λ_3
1	-54.1994	-1.6917 -13.4095i	-1.6917 +13.4095i
0.9	-55.6819	-0.8162 -13.4846i	-0.8162 +13.4846i
0.8	-57.4199	0.2206 -13.5224i	0.2206 +13.5224i
0.7	-59.4985	1.4758 -13.4936i	1.4758 +13.4936i

Table 2: Closed loop eigenvalues vs. p for the more robust design

5. Eigenvalue Assignment: Exercise 6.9 from the book.

Solution:

We follow the same procedure as in Example 6.4 of the textbook to find the characteristic polynomial of the closed loop system. The closed loop system can be found to be

$$\frac{dx}{dt} = \begin{bmatrix} -k_1 & 1 - k_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k_r \\ 0 \end{bmatrix} r$$

The characteristic polynomial of the closed loop system is given by

$$det \begin{bmatrix} \lambda + k_1 & -1 + k_2 \\ 0 & \lambda \end{bmatrix} = \lambda^2 + k_1 \lambda$$

This gives the eigenvalues $\lambda = 0, -k_1$. So, whatever be the values of k_1, k_2 one eigenvalue, 0, is fixed. Thus, the eigenvalues cannot be assigned arbitrary values.

6. Second Order Systems and State Feedback Design: Thus far, we have largely focused on stability as our performance metric of choice. Yet once stability is assured, we can focus on finer details. A few classes ago, we discussed other system properties, such as rise time, overshoot, and settling time. In this exercise, we will explore how controlling eigenvalues through Eigenvalue assignment is important beyond just placing them in the OLHP to insure stability.

   Read the beginning of Section 6.3 on second order systems. Replicate the computations in Example 6.6 exploring the tradeoff of overshoot and rise time by plotting the curves for smaller values of ζ (the book shows the curve for $\zeta = 0.9$).

   Solution: left to students.

7. (Optional) Cayley-Hamilton Theorem: Exercise 6.10 from the book.

   Solution: left to students.