The University of Texas at Austin
Department of Electrical and Computer Engineering

EE362K: Introduction to Automatic Control—Fall 2009

Problem Set Seven

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This problem set is intended to get us started thinking about differential equations and their solution, as well as properties of the solution.

1. Computing Laplace Transforms: Compute the Laplace transform of each of the following:
   - $f(t) = \int_0^t \cos(t-\tau)\sin(\tau)\,d\tau$.
   - For $f(t)$ as given in the first part of this problem, define the function
   $$h(t) = \int_0^t \int_0^{t_1} \int_0^{t_2} f(\tau)\,d\tau\,dt_1\,dt_2.$$
   Find $\mathcal{L}(h)$.
   - $f(t) = (\sin t)/t$.

2. Inverse Laplace transform, and partial fraction expansions: If you have not seen these techniques in previous classes, you may want to look here: http://en.wikipedia.org/wiki/Partial_fraction. The relevant section of this article are: (1) the introduction, (2) Distinct first-degree factors in the denominator, (3) A repeated first-degree factor in the denominator, (4) Repeated factors in the denominator generally. The remaining sections are interesting as well, but not necessary for solving the homework problems.
   - $H(s) = \frac{2(s+2)}{(s+1)(s^2+4)}$;
   - $H(s) = \frac{s-2}{(s+3)^2(s+2)^2}$; (Note the repeated roots here).

3. All the transfer functions we will see in this class are rational (ratios of polynomials) where the degree of the numerator is at most the degree of the denominator. As we discussed in class, the stability is determined by the poles of the transfer function, that is, the zeros of the denominator. There is a classical technique known as the Routh-Hurwitz stability criterion, which allows us to determine whether the roots of a polynomial lie in the left half of the complex plane without explicitly computing the roots.[1] This is a simple test, and it goes as follows: Consider a polynomial
$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n.$$

[1] For degree two equations, there is the well-known quadratic equation for solving for the roots. Similar expressions exist for third and fourth order polynomials. It was a long-standing open problem to find a similar formula for fifth order polynomials. In 1832, Évariste Galois proved that *no such expression can exist.* In particular, he proved that there must exist polynomials of order five, whose roots *are not expressible by radicals.* Surely, some fifth order polynomials do have this property, e.g., $x^5 - 2$, as one of its roots is $\sqrt[n]{2}$. On the other hand, the polynomial $p(x) = x^5 - x - 1$ has roots which are not expressible by radicals.

(Note that we can always normalize the polynomial coefficients so that the highest order coefficient is 1, without changing the location of the polynomial roots.) We form the *Routh Array* as follows:

$$
\begin{array}{llllll}
\text{Sequence 1:} & 1 & a_2 & a_4 & a_6 & \ldots \\
\text{Sequence 2:} & a_1 & a_3 & a_5 & a_7 & \ldots \\
\text{Sequence 3:} & b_1 & b_2 & b_3 & b_4 & \ldots \\
\text{Sequence 4:} & c_1 & c_2 & c_3 & c_4 & \ldots \\
\text{Sequence 5:} & d_1 & d_2 & d_3 & d_4 & \ldots \\
\vdots & & \vdots & & \vdots &
\end{array}
$$

The first and second sequences are formed from the even and odd coefficients of the polynomial. All subsequent sequences are formed from the two sequences directly above as illustrated below:

$$
\begin{array}{llll}
b_1 = \frac{a_1 a_2 - a_3}{a_1}, & b_2 = \frac{a_1 a_4 - a_5}{a_1}, & b_3 = \frac{a_1 a_6 - a_7}{a_1} & \cdots \\
c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, & c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}, & c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1} & \cdots \\
d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}, & d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}, & d_3 = \frac{c_1 b_4 - b_1 c_4}{c_1} & \cdots
\end{array}
$$

The array terminates because later rows have successively fewer entries, and finally we have two successive rows with just a single entry. The Routh-Hurwitz test asserts that the roots are all in the left hand plane, as long as the coefficients in the first column of the Routh array are all positive.

(a) Suppose the denominator of the transfer function has the form:

$$1 + KG = s^4 + 2s^3 + 3s^2 + 8s + 8.$$

Use the test above to show that this system is not stable.

(b) Consider a a transfer function with denominator of the form

$$q(s) = s^3 + 5s^2 + (K - 6)s + K.$$

Use the Routh-Hurwitz test to determine the range of K for which the system is stable.

(c) Consider the block diagram below, where the plant G is given by

$$G(s) = \frac{1}{s^2 + 3s - 4},$$

and the controller D is a PID controller given by:

$$D(s) = K_1 + \frac{1}{s} K_2 + s K_3.$$

- Compute the closed loop transfer function from r to y.
- Compute conditions on the constants K_1, K_2, and K_3, so that the resulting closed loop system is stable.

4. Simplify the following block diagrams to determine the single transfer function from the input to output.

   (a) Block Diagram 1:

   (b) Block Diagram 2: