The University of Texas at Austin
Department of Electrical and Computer Engineering

EE380K: Linear Systems Theory—Fall 2006

Midterm Exam One

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I. This is a closed-book, closed-notes exam. You do not need a calculator, and you will not need any special formulas.

II. Please be clear about any assumptions you make. Make it easy for me to give you partial credit.

III. Do all your work in the pages provided. If you need extra paper, ask for it. Make sure to put your name on all loose pages.

IV. There are 5 problems, for a total of 105 points. The breakdown of the point values are as follows (so divide your effort accordingly, i.e., as you see fit):

1. 20%
2. 20%
3. 20%
4. 15%
5. 30%.

Problem 1 (20%) (a) (4%) Find the Jordan canonical form (i.e., the matrix J, you do not need to find the matrix S) for each of the matrices below:

$$A_1 = \begin{bmatrix} 8 & 23 & 4 & 4 \\ 0 & -12 & -1 & 2 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 8 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(b) (5%) Find the Jordan canonical form for the matrix below, *and the basis for this representation.*

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(c) (3%) Is the autonomous system $\dot{x} = -A_3 x$ asymptotically stable, marginally stable, or unstable?

(d) (4%) Prove that the following two matrices are not similar, (without appealing to the Jordan canonical form) that is, there does not exist an invertible matrix S such that $A = S^{-1}BS$.

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(e) (4%) Prove that the following two matrices are not similar:

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Problem 2 (20%) (a) (10%) Consider a matrix $A \in \mathbb{R}^{m \times n}$. The row-rank of A is defined as the number of independent rows, and the column-rank of A is defined as the number of independent columns.

Show that the row-rank is always equal to the column rank. Show this directly (so, e.g., do not just quote the SVD theorem to conclude the result). (Hint: think range and null space).

(b) (10%) Suppose we have linear operators:

$$\begin{aligned} A &: V \to W \\ B &: V \to W \\ T &: W \to U. \end{aligned}$$

Under what conditions on T, does $TA = TB$ imply that $A = B$?

Problem 3 (20%) Consider a discrete time linear time-invariant system:

$$x(k+t) = Ax(k) + Bu(k).$$

Given an initial state, $x(0) = x_0$, a termination time K, and a desired final state, $x(K) = x_K$, we want to find the smoothest control law that will take the system from x_0 to x_K in K time steps. That is, we want to solve the problem:

$$\begin{aligned} \text{min}: &\quad \tfrac{1}{K}\left(\textstyle\sum_{k=0}^{K-1}(u(k) - u(k-1))^2\right)^{1/2} \\ \text{s.t.}: &\quad x(K) = x_K. \end{aligned}$$

(Assume $u(-1) = 0$). Answer the following:

(a) (8%) Give a sufficient condition for this problem to have a solution (i.e., regardless of the objective function).

(b) (12 %) Formulate the problem of finding the smoothest control law, and give an expression for the solution.

Two hints:

- Least squares.
- Recall that the solution to the discrete time LTI system is:

$$x(k) = A^k x(0) + \sum_{l=0}^{k-1} A^{k-l-1} Bu(l).$$

Problem 4 (15%) Consider the three discrete-time autonomous linear systems:

$$\begin{aligned} x_1(k+1) &= A_1 x(k) \\ x_2(k+1) &= A_2 x(k) \\ x_3(k+1) &= A_{12}(k) x(k), \end{aligned}$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \\ A_2 &= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} \\ A_{12}(k) &= \begin{cases} A_1 & \text{if } k \text{ is even,} \\ A_2 & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

(a) (5%) Give conditions on (a_{ij}) and (b_{ij}) that guarantee that $x_1(k) \to 0$ and $x_2(k) \to 0$, as $k \to \infty$.

(b) (5%) Give an example where $||x_1(k)|| \to \infty$, and $||x_2(k)|| \to \infty$, as $k \to \infty$, but $||x_3(k)|| \to 0$ as $k \to \infty$.

(c) (5%) Give necessary and sufficient conditions for $||x_3(k)|| \to 0$ as $k \to \infty$.

Problem 5 (30%) Consider the perturbed system:

$$q^{(3)} + (3 + \delta_2)q^{(2)} + (3 + \delta_1)q^{(1)} + (1 + \delta_0)q = u.$$

(a) (5 %) Find a state-space realization for this system:

$$\dot{x} = Ax + Bu.$$

(b) (5 %) If $\boldsymbol{\delta} = (0, 0, 0)$, is this system asymptotically stable?

(c) (5 %) The matrix A can be written as $A = S^{-1}JS$ where J is in Jordan canonical form. Find J, for $\boldsymbol{\delta} = \mathbf{0}$.

(d) (15 %) Find the smallest perturbation $||\delta||_2$ (this is the usual 2-norm for vectors) such that the system has a pole at $j\omega$, *for some fixed* $\omega \in \mathbb{R}$.