The University of Texas at Austin
Department of Electrical and Computer Engineering

EE380K: Linear Systems Theory—Fall 2008

PROBLEM SET ZERO

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This problem set is intended to get the semester off to a good start (!) and to help you refresh your memory about basic concepts of linear algebra. Any problems marked by a ($*$) are optional (bonus) problems.

Linear Algebra

1. Vector Spaces: For the following examples, state whether or not they are in fact vector spaces.
   - The set of polynomials in one variable, of degree at most d.
   - The set of continuous functions mapping $[0, 1]$ to $[0, 1]$, such that $f(0) = 0$.
   - The set of continuous functions mapping $[0, 1]$ to $[0, 1]$, such that $f(1) = 1$.
2. Recall that in class we defined a linear operator $T : V \to W$ to be a map that satisfies:
$$T(a\boldsymbol{v}_1 + b\boldsymbol{v}_2) = aT\boldsymbol{v}_1 + bT\boldsymbol{v}_2,$$
   for every $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$.

   Show which of the following maps are linear operators:
   - $T : V \to V$ given by the identity map: $\boldsymbol{v} \mapsto \boldsymbol{v}$.
   - $T : V \to W$ given by the constant map: $\boldsymbol{v} \mapsto \boldsymbol{w}_0$ for every $\boldsymbol{v} \in V$. Does your answer change depending on what $\boldsymbol{w}_0$ is?
   - Let V be the vector space of polynomials of degree at most d. Let $T : V \to V$ be the map defined by the derivative: $p(x) \mapsto p'(x)$.
   - For V as above, let T be given by:
$$T(p) = \int_0^1 p(x)\, dx.$$
   - What about
$$T(p) = \int_0^1 p(x)x^3\, dx.$$
3. Independence:
   - If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in V$ are independent, and $T : V \to W$ is a linear operator, is it true that $T\boldsymbol{v}_1, \ldots, T\boldsymbol{v}_m \in W$ are independent?

- If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in V$ are dependent, and $T : V \to W$ is a linear operator, is it true that $T\boldsymbol{v}_1, \ldots, T\boldsymbol{v}_m \in W$ are dependent?

4. True or False: If vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are elements of a vector space V, and $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, $\{\boldsymbol{v}_2, \boldsymbol{v}_3\}$, and $\{\boldsymbol{v}_1, \boldsymbol{v}_3\}$ are independent, then the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is also linearly independent.

5. Range and Nullspace of Matrices: Recall from class the definition of the null space and the range of a linear transformation, $T : V \to W$:

$$\begin{aligned} \text{null}T &= \{\boldsymbol{v} \in V \, : \, T\boldsymbol{v} = 0\} \\ \text{range}T &= \{T\boldsymbol{v} \in W \, : \, \boldsymbol{v} \in V\} \end{aligned}$$

   - Suppose A is a 10-by-10 matrix of rank 5, and B is also a 10-by-10 matrix of rank 5. What is the **smallest** and **largest** the rank the matrix $C = AB$ could be?
   - Now suppose A is a 10-by-15 matrix of rank 7, and B is a 15-by-11 matrix of rank 8. What is the **largest** that the rank of matrix $C = AB$ can be?

6. Riesz Representation Theorem: Consider the standard basis for $\mathbb{R}^n$: $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0)$, etc. Recall that the inner-product of two vectors $\boldsymbol{w}_1 = (\alpha_1, \ldots, \alpha_n), \boldsymbol{w}_2 = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, is given by:

$$\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle = \sum_{i=1}^{n} \alpha_i \beta_i.$$

   Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear map. Show that there exists a vector $\boldsymbol{x} \in \mathbb{R}^n$, such that

$$f(\boldsymbol{w}) = \langle \boldsymbol{x}, \boldsymbol{w} \rangle,$$

   for any $\boldsymbol{w} \in \mathbb{R}^n$.

   It turns out that this result is true in much more generality. For example, consider the vector space of square-integrable functions (something we will see much more later in the course). Let F denote a linear map from square integrable functions to $\mathbb{R}$. Then, as a consequence similar to the finite dimensional exercise here, there exists a square integrable function, g, such that:

$$F(f) = \int fg.$$

7. Let V be the vector space of (univariate) polynomials of degree at most d. Consider the mapping $T : V \to V$ given by:

$$Tp = a_0 p(t) + a_1 t p^{(1)}(t) + a_2 t^2 p^{(2)}(t) + \cdots + a_d t^d p^{(d)}(t),$$

   where $p^{(r)}(t)$ denotes the r^{th} derivative of the polynomial p.

   - True or False: if $Tp = 2p(t) - tp'(t)$, then for every polynomial $q \in V$, there exists a polynomial $p \in V$, with $Tp = q$.
   - What about for T given by $Tp = 2p(t) - 3tp'(t)$?
   - Provide a characterization of the set of coefficients $(a_0, a_1, \ldots, a_d)$, such that the operator T they define has the property that for every polynomial $q \in V$, there exists a polynomial $p \in V$, with $Tp = q$.

8. (∗) Consider a mapping $T : V \to V$. If the vector space V is finite dimensional, then if null$T = \{0\}$, T is surjective (also known as onto), that is, for any $\boldsymbol{v} \in V$, there exists $\hat{\boldsymbol{v}}$ such that $T\hat{\boldsymbol{v}} = \boldsymbol{v}$. Conversely, if T is surjective, then null$T = \{0\}$, and $T\boldsymbol{v} = 0$ implies $\boldsymbol{v} = 0$.

   - Give an example of an infinite dimensional vector space, V, and a linear operator $T : V \to V$, such that T is surjective, but null$T \neq \{0\}$.
   - Give an example of an infinite dimensional vector space, V, and a linear operator $T : V \to V$, such that null$T = \{0\}$, but T is not surjective.

   [Hint: consider the space of polynomials of arbitrary degree.]