The University of Texas at Austin
Department of Electrical and Computer Engineering

EE380K: Linear Systems Theory—Fall 2008

SOLUTIONS FOR PROBLEM SET ZERO

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1. Vector Spaces

   - The set of polynomials in one variable, of degree at most d is a vector space:
     Closed under addition and scalar multiplication:

     $$\alpha(a_0+a_1x^1+...+a_dx^d)+(b_0+b_1x^1+...+b_dx^d) = (\alpha a_0+b_0)+(\alpha a_1+b_1)x^1+...+(\alpha a_d+b_d)x^d$$

     Zero vector: 0 is a polynomial of degree less than d.
     Other properties follow naturally.

   - $\hat{S}$ = the set of continuous functions mapping $[0, 1]$ to $[0, 1]$, such that $f(0) = 0$ is not a vector space: Consider $f_1(x) = f_2(x) = x$. $f_1(1)+f_2(1) = x+x = 2x$, so $f_1(1)+f_2(1) \notin \hat{S}$ because it maps $[0, 1]$ onto $[0, 2]$.
   - $\hat{S}$ = the set of continuous functions mapping $[0, 1]$ to $[0, 1]$, such that $f(1) = 1$ is not a vector space: Consider $f_1(x) = f_2(x) = 1$. $f_1(1)+f_2(1) = 1+1 = 2$, so $f_1(1)+f_2(1) \notin \hat{S}$.

2. Show which of the following maps are linear operators:

   - $T : V \to V$ given by the identity map: $\boldsymbol{v} \mapsto \boldsymbol{v}$.
     For every $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$,

     $$T(a\boldsymbol{v}_1 + b\boldsymbol{v}_2) = a\boldsymbol{v}_1 + b\boldsymbol{v}_2 = aT\boldsymbol{v}_1 + bT\boldsymbol{v}_2.$$

     Linear.
   - $T : V \to W$ given by the constant map: $\boldsymbol{v} \mapsto \boldsymbol{w}_0$ for every $\boldsymbol{v} \in V$. If $\boldsymbol{w}_0 = 0$, then for every $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$,

     $$T(a\boldsymbol{v}_1 + b\boldsymbol{v}_2) = 0 = aT\boldsymbol{v}_1 + bT\boldsymbol{v}_2.$$

     Linear.
     If $\boldsymbol{w}_0 \neq 0$, then for any $\boldsymbol{v}_1 \in V$,

     $$T(\boldsymbol{v}_1 + \boldsymbol{v}_1) = \boldsymbol{w}_0 \neq \boldsymbol{w}_0 + \boldsymbol{w}_0 = T\boldsymbol{v}_1 + T\boldsymbol{v}_1.$$

     Not Linear.

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[1] Many of the solutions are in whole or in part written by Johnson Carroll.

- Let V be the vector space of polynomials of degree at most d. Let $T : V \to V$ be the map defined by the derivative: $p(x) \mapsto p'(x)$.

  $$\begin{aligned}&T(\alpha(a_0 + a_1x + ... + a_dx^d) + \beta(b_0 + b_1x + ... + b_dx^d))\\ &= \alpha(0 + a_1 + ... + a_dx^{d-1}) + \beta(0 + b_1 + ... + b_dx^{d-1})\\ &= \alpha T(a_0 + a_1x + ... + a_dx^d) + \beta T(b_0 + b_1x + ... + b_dx^d)\end{aligned}$$

  Linear.

- For V as above, let T be given by:

  $$T(p) = \int_0^1 p(x)\, dx.$$

  $$\begin{aligned}&T(\alpha(a_0 + a_1x + ... + a_dx^d) + \beta(b_0 + b_1x + ... + b_dx^d))\\ &= \alpha a_0x + \alpha a_1x^2/2 + ... + \alpha a_dx^{d+1}/(d + 1) + \beta b_0x + \beta b_1x^2/2 + ... + \beta b_dx^{d+1}/(d + 1)|_0^1\\ &= \alpha a_0 + \alpha a_1/2 + ... + \alpha a_d/(d + 1) + \beta b_0 + \beta b_1/2 + ... + \beta b_d/(d + 1)\\ &= \alpha(a_0x + a_1x^2/2 + ... + a_dx^{d+1}/(d + 1)) + \beta(b_0x + b_1x^2/2 + ... + b_dx^{d+1}/(d + 1))|_0^1\\ &= \alpha T(a_0 + a_1x + ... + a_dx^d) + \beta T(b_0 + b_1x + ... + b_dx^d)\end{aligned}$$

  Linear.

- What about

  $$T(p) = \int_0^1 p(x)x^3\, dx.$$

  $$\begin{aligned}&T(\alpha(a_0 + a_1x + ... + a_dx^d) + \beta(b_0 + b_1x + ... + b_dx^d))\\ &= \alpha a_0x^4/4 + \alpha a_1x^5/5 + ... + \alpha a_dx^{d+4}/(d + 4) + \beta b_0x^4/4 + \beta b_1x^5/5 + ... + \beta b_dx^{d+4}/(d + 4)|_0^1\\ &= \alpha a_0/4 + \alpha a_1/5 + ... + \alpha a_d/(d + 4) + \beta b_0/4 + \beta b_1/5 + ... + \beta b_d/(d + 4)\\ &= \alpha(a_0x^4/4 + a_1x^5/5 + ... + a_dx^{d+4}/(d + 4)) + \beta(b_0x^4/4 + b_1x^5/5 + ... + b_dx^{d+4}/(d + 4))|_0^1\\ &= \alpha T(a_0 + a_1x + ... + a_dx^d) + \beta T(b_0 + b_1x + ... + b_dx^d)\end{aligned}$$

  Linear.

- Note that, in general, integration against any function is a linear map. If the operator T is defined as $T(f) = \int fg$, then if $f = \alpha f_1 + \beta f_2$, we have:

  $$\begin{aligned}T(f) &= \int f(x)g(x)\, dx\\ &= \int(\alpha f_1(x) + \beta f_2(x))g(x)\, dx\\ &= \alpha \int f_1(x)g(x)\, dx + \beta \int f_2(x)g(x)\, dx\\ &= \alpha T(f_1) + \beta T(f_2).\end{aligned}$$

3. Independence:

   - $T\boldsymbol{v} = \mathbf{0}$ is a linear map, but any pair of independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ are both mapped to $\mathbf{0}$, meaning that $T\boldsymbol{v}_1, T\boldsymbol{v}_2$ are dependent.
   - Suppose (without loss of generality) that $\boldsymbol{v}_m = \sum_{i=1}^{m-1} a_i \boldsymbol{v}_i$. $T\boldsymbol{v}_m = T(\sum_{i=1}^{m-1} a_i \boldsymbol{v}_i) = \sum_{i=1}^{m-1} a_i T\boldsymbol{v}_i$, so $\{T\boldsymbol{v}_i\}$ are dependent.

4. False. Consider $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

5. Range and Nullspace of Matrices:

   - $0 \leq rank(AB) \leq 5$.
   - $rank(AB) \leq 7$.

6. Riesz Representation Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear map, and let $\boldsymbol{w}$ be an arbitrary element of $\mathbb{R}^n$.

   $$f(\boldsymbol{w}) = f(\sum_{i=1}^{n} w_i \boldsymbol{e}_i) = \sum_{i=1}^{n} f(w_i \boldsymbol{e}_i) = \sum_{i=1}^{n} f(\boldsymbol{e}_i) w_i = \langle f(\boldsymbol{e}_i), \boldsymbol{w} \rangle.$$

   $f(\boldsymbol{e}_i)$ is the vector we sought, and therefore the theorem is proved.

7. Let V be the vector space of (univariate) polynomials of degree at most d. Consider the mapping $T : V \to V$ given by:

   $$Tp = a_0 p(t) + a_1 t p^{(1)}(t) + a_2 t^2 p^{(2)}(t) + \cdots + a_d t^d p^{(d)}(t),$$

   where $p^{(r)}(t)$ denotes the r^{th} derivative of the polynomial p.

   - True or False: if $Tp = 2p(t) - tp'(t)$, then for every polynomial $q \in V$, there exists a polynomial $p \in V$, with $Tp = q$.
     (*Added Note: V still consists of polynomials of degree at most d, where d is some fixed but arbitrary number, i.e., d need not be equal to 1*)
     False. $x^2 \in \text{Null}T$. That is, for any choice of p, the t^2 term of Tp has a coefficient 0.
   - What about for T given by $Tp = 2p(t) - 3tp'(t)$? True. Note that the basis $(1, t, t^2, ..., t^d)$ is mapped to $(2, -t, -4t^2, ..., (2-3k)t^k, ..., (2-3d)t^d)$ which is independent, and hence it is surjective.
   - Provide a characterization of the set of coefficients $(a_0, a_1, \ldots, a_d)$, such that the operator T they define has the property that for every polynomial $q \in V$, there exists a polynomial $p \in V$, with $Tp = q$.
     Let the ith coefficient of p be called p_i.

     $$q_i = (\sum_{j=0}^{i} a_i \frac{j!}{(j-i)!}) \cdot p_i.$$

     So if

     $$\sum_{j=0}^{i} a_i \frac{j!}{(j-i)!} \neq 0,$$

     for all $i = 0, 1, 2, ..., d$, then the property is satisfied.

8.

- Consider (as the hint suggests) the space of polynomials of arbitrary degree, and the derivative map examined earlier. Any constant maps to $\mathbf{0}$, so $\text{null}T \neq \{0\}$, but the map is surjective.
- Consider polynomials of arbitrary degree, and let $Tp(t) = t \cdot p(t)$. T is linear:

  $$T(\alpha p(t) + \beta q(t)) = \alpha t p(t) + \beta t q(t) = \alpha T(p(t)) + \beta T(q(t)),$$

  and $\text{null}T = \{0\}$. However, there is no $\boldsymbol{v} \in V$ such that $T\boldsymbol{v}$ is constant non-zero, so T is not surjective.