

**EE380K: Linear Systems Theory—Fall 2008**

PROBLEM SET ONE

C. Caramanis

Not Due: For Practice and Review Only.

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This problem set is intended to give you practice with linear algebra, including both matrix theory, and also the theory of linear operators.

1. Eigenvalues and Eigenvectors. Recall from class that we defined an eigenvector of a linear operator  $T : V \rightarrow V$  to be a (nonzero) vector  $v$  such that:

$$Tv = \lambda v,$$

for some scalar  $\lambda$ . The scalar  $\lambda$  is called the eigenvalue of  $T$  corresponding to the eigenvector  $v$ .

- (a) Use matlab to compute the eigenvectors and eigenvalues of the matrix:

$$A = \begin{pmatrix} 2 & 6 & 7 \\ 6 & 12 & 6 \\ 7 & 6 & 4 \end{pmatrix}.$$

- (b) Generate a random  $3 \times 3$  matrix,  $T$ , and compute the eigenvectors and eigenvalues of  $T^{-1}AT$ . What do you notice? Now prove that for any matrix  $A$ , and any invertible matrix  $T$ , the set of eigenvalues of  $A$  and  $T^{-1}AT$  must be the same.

Note that this shows that the notion of eigenvalue is independent of the basis representation we choose for a linear operator. Thus we can speak of the eigenvalues of a linear operator without reference to a particular matrix.

- (c) For  $A$  the matrix given above, and  $T$  the random matrix you generated, compute (i) The trace<sup>1</sup> of  $A$ ; (ii) The trace of  $T^{-1}AT$ , and (iii) the sum of the eigenvalues of  $A$ .

2. Let  $P_d$  denote the space of univariate (at most) degree- $d$  polynomials.

- (a) Let  $T : P_d \rightarrow P_d$  be given by  $Tp = p'$ . Compute the matrix corresponding to  $T$ , using the basis  $\{1, t, t^2, \dots, t^d\}$ .
- (b) Compute the eigenvalues of  $T$ . Also, compute an eigenvector of  $T$  (computing one is enough for this exercise).
- (c) Let  $T : P_d \rightarrow P_d$  be the map given by:

$$T(p) = 4p^{(3)} + 2p^{(2)} - 3p',$$

(where  $p^{(k)}$  denotes the  $k^{\text{th}}$  derivative). Compute the determinant of  $T$ .

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<sup>1</sup>The trace of a matrix is the sum of the diagonal elements:  $\text{Trace}(A) = \sum_i A_{ii}$ .

3. Consider two points,  $v_1, v_2 \in \mathbb{R}^n$ . Show that there exist  $c \in \mathbb{R}^n$  and  $d \in \mathbb{R}$  (and find them!) such that

$$\{x : \|x - v_1\| \leq \|x - v_2\|\} = \{x : c^\top x \leq d\}.$$

Thus, you are showing that the set of points in  $\mathbb{R}^n$  that are closer to point  $v_1$  than to point  $v_2$ , form a half-space.

4. Let  $A$  be an  $n \times m$  real matrix, and  $B$  a  $k \times m$  real matrix. Suppose that for every  $x \in \mathbb{R}^m$ ,  $Ax = 0$  only if  $Bx = 0$ , that is,

$$Ax = 0 \Rightarrow Bx = 0.$$

Show that there exists a  $k \times n$  real matrix  $C$  such that  $CA = B$ .