The University of Texas at Austin
Department of Electrical and Computer Engineering

EE380K: Linear Systems Theory—Fall 2008

Problem Set 10

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Here are some practice problems relating to the last part of the class, including the small gain theorem.

1. Exercise 19.2 from the course notes.
2. Exercise 19.3 from the course notes.
3. Exercise 19.4 from the course notes.
4. Exercise 19.5 from the course notes.
5. Consider a SISO CT system, with nominally strictly proper plant transfer function:

   $$P(s) = \frac{m(s) + \delta m(s)}{d(s) + \delta d(s)},$$

   where $m(s)$ and $d(s)$ are polynomials:

   $$\begin{aligned} m(s) &= m_{n-1}s^{n-1} + \cdots + m_1 s + m_0 \\ d(s) &= s^n + d_{n-1}s^{n-1} + \cdots + d_0. \end{aligned}$$

   Assume that $m(s)$ and $d(s)$ have no common factors. The perturbation polynomials $\delta m(s)$ and $\delta d(s)$ are given by:

   $$\begin{aligned} \delta m(s) &= r_{n-1}s^{n-1} + \cdots + r_1 s + r_0 \\ \delta d(s) &= t_{n-1}s^{n-1} + \cdots + t_0. \end{aligned}$$

   Suppose we have a constant output feedback gain, k. (So the block diagram has the controller block K in the feedback loop). Assume that this constant gain controller is such that it stabilizes the nominal plant.

   Compute the minimal 2-norm of the perturbation polynomials $\delta m(s)$ and $\delta d(s)$ required to make the resulting closed loop system unstable.

   More specifically, compute:

   $$\min_{r,t}\{||r||_2^2 + ||t||_2^2 \,|\, \text{closed loop system unstable.}\}.$$

   [The final answer should be expressed as:

   $$\begin{aligned} \min: &\quad F(\omega) \\ \text{s.t.}: &\quad \omega \in \mathbb{R}, \end{aligned}$$

   where F is an explicit function you can write down.]

6. Exercise 20.1 from the course notes. For the first part, consider first writing the condition that W must satisfy (you can find the relevant discussion at the beginning of Chapter 19). It turns out that $\tilde{\alpha} \notin [0.5, 1.5]$ (remember that W must be stable), which is fine, since all we want is an uncertainty set that contains the given uncertainty set.

7. (a) Let the open-loop transfer function G be given in the following block form:

$$G(s) = \begin{bmatrix} T_{\psi\theta}(s) & T_{\psi w}(s) \\ T_{z\theta}(s) & T_{zw}(s) \end{bmatrix}.$$

   Compute the closed-loop transfer function from w to z (that is, when we have $\theta = \Delta\psi$).

   (b) Conclude one direction of the small gain theorem, namely, that $||T_{\psi\theta}||_\infty < 1$ is a sufficient condition for the closed loop transfer function computed in the previous part to be stable, for all $||\Delta||_\infty \leq 1$. Assume that the four open-loop transfer functions given above are stable.

8. (a) Consider a servo feedback system as illustrated. Suppose we have uncertainty in the plant P, of the form:

$$P \in \Omega_P \triangleq \{P \,:\, P = (I - W_P\Delta)^{-1}P_0, \ ||\Delta||_\infty \leq 1\}.$$

   Draw the plant diagram representing this plant error.

   (b) Compute the stability robustness condition in terms of K, P_0, and W.

   (c) Now suppose we know the plant exactly, i.e., $P = P_0$, but consider a disturbance $d = W_d\xi$ added after the plant, as in the diagram below. Derive the transfer function from the noise ξ to the output, y.

   (d) Realize the disturbance of the previous problem as plant uncertainty. That is, draw a model of plant uncertainty so that the resulting stability robustness condition is exactly the same as the disturbance rejection condition $||T_{y\xi}||_\infty < 1$, for $T_{y\xi}$ the transfer function you computed in the previous part.

   (e) In addition to the disturbance, suppose we have plant uncertainty, of the form:

$$P \in \Omega_P \triangleq \{P \,:\, P = P_0(I + W_P\Delta), \ ||\Delta||_\infty \leq 1\}.$$

   (Note that this is different from the plant uncertainty given above.) If the system is SISO, compute a sufficient condition for **Robust Disturbance Rejection**, i.e., compute a sufficient condition that will guarantee that the system remain stable for any $P \in \Omega_P$, and also that the disturbance will be rejected.