The University of Texas at Austin
Department of Electrical and Computer Engineering

EE380K: Linear Systems Theory—Fall 2008

SOLUTIONS FOR PROBLEM SET ONE

Constantine Caramanis[1] Due: Wednesday, September 10, 2008.

---

1. Eigenvalues and Eigenvectors:

$$A = \begin{bmatrix} 2 & 6 & 7 \\ 6 & 12 & 6 \\ 7 & 6 & 4 \end{bmatrix}$$

(a) Eigenvalue/Eigenvector pairs:

$$-4.1038, \begin{bmatrix} 0.7752 \\ -0.0543 \\ -0.6294 \end{bmatrix}; 2.5410, \begin{bmatrix} 0.4430 \\ -0.6635 \\ 0.6029 \end{bmatrix}; 19.5628, \begin{bmatrix} 0.4503 \\ 0.7462 \\ 0.4903 \end{bmatrix};$$

(b) For my random T,

$$T^{-1}AT = \begin{bmatrix} 1.4570 & 6.3906 & 4.2811 \\ 13.2279 & 18.9085 & 7.7904 \\ -0.6087 & -7.9013 & -2.3655 \end{bmatrix}$$

Eigenvalue/Eigenvector pairs:

$$19.5628, \begin{bmatrix} -0.2422 \\ -0.9106 \\ 0.3348 \end{bmatrix}; -4.1038, \begin{bmatrix} -0.7315 \\ 0.1998 \\ 0.6519 \end{bmatrix}; 2.5410, \begin{bmatrix} 0.1650 \\ -0.5295 \\ 0.8321 \end{bmatrix};$$

Next we have to show that this is generically true, namely, that eigenvalues are invariant under similarity transformations. Again, this is significant, because it tells us that the notion of eigenvalue is something inherent to the linear operator, and not the basis which we choose to represent it by. The matrix itself, for instance, is of course dependent on the basis we choose.

Consider then, an eigenvalue λ of the matrix A; let v denote the eigenvector, so $Av = \lambda v$. Then, note that:

$$\begin{aligned} T^{-1}AT(T^{-1}v) &= T^{-1}Av \\ &= T^{-1}(\lambda v) \\ &= \lambda(T^{-1}v). \end{aligned}$$

---

[1]Many of the solutions written in whole or in part by Johnson Carroll.

Therefore A and $T^{-1}AT$ have the same eigenvalues, and in fact we also understand how to relate the eigenvectors of A with the eigenvectors of $T^{-1}AT$.

(c) Again using matlab,

$$\begin{aligned} trace(A) &= 18 \\ trace(T^{-1}AT) &= 18.0000 \\ \text{sum of eigenvalues of } A &= 18.0000 \end{aligned}$$

Note the three expressions are equal. It is in fact true that the trace is also invariant to similarity transformations. But, here is a bonus question, which we will understand better in the coming lectures: is the trace always equal to the sum of the eigenvalues??

2. Let P_d be the space of univariate (at most) degree-d polynomials.

(a) Let $T : P_d \to P_d$ be given by $Tp = p'$. Compute the matrix corresponding to T, using the basis $\{1, t, t^2, ..., t^d\}$.

Remember from class that the $(i, j)^{th}$ element of the matrix corresponding to T is given by

$$Tv_j = \sum_{i=1}^{d+1} a_{ij} v_i, \text{ where } \{v_j\}_{i=1}^{d-1} \text{ is a basis for V.}$$

Applying this formula to the transformation and basis provided yields the following matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 2 & 0 & & & \\ 0 & 0 & 0 & 3 & & & \\ \cdot & & & & \cdot & & \\ \cdot & & & & & \cdot & \\ \cdot & & & & & & \cdot \end{bmatrix}$$

(b) The eigenvalues of T are the eigenvalues of the matrix: all zeros. Because the only eigenvalue is 0, the eigenvectors are polynomials in the null space of T ($Tv = 0 \cdot v$). For the derivative map, the eigenvectors are the constants (polynomials of degree 0). For any finite d, this operator (and matrix) is *nilpotent*. A (nonzero) operator T is called nilpotent if for some integer n, T^n is the zero operator (note that no scalar has this property). And indeed, in the space of degree d polynomials, the mapping given by T^{d+1}, i.e., the $(d + 1)^{\text{st}}$ derivative, is always identically zero.

(c) Let $T : P_d \to P_d$ be given by $Tp = 4p^{(3)} + 2p^{(2)} - 3p'$. Compute the determinant of T. Note that T is not 1-1 (injective), since $T(4) = T(-2) = 0$. Since T is not 1-1, T is not invertible, and therefore the determinant of T must be 0.

3. Consider two points, $v_1, v_2 \in \mathbb{R}^n$. Show that there exist $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ (and find them!) such that

$$\{x : \|x - v_1\| \le \|x - v_2\|\} = \{x : c^T x \le d\}.$$

$$
\begin{aligned}
x \in \{x : \|x - v_1\| \leq \|x - v_2\|\} &\Leftrightarrow \|x - v_1\| \leq \|x - v_2\| \\
&\Leftrightarrow \|x - v_1\|^2 \leq \|x - v_2\|^2 \\
&\Leftrightarrow \langle x - v_1, x - v_1 \rangle \leq \langle x - v_2, x - v_2 \rangle \\
&\Leftrightarrow \langle x, x \rangle - 2\langle v_1, x \rangle + \langle v_1, v_1 \rangle \leq \langle x, x \rangle - 2\langle v_2, x \rangle + \langle v_2, v_2 \rangle \\
&\Leftrightarrow 2\langle v_2, x \rangle - 2\langle v_1, x \rangle \leq \langle v_2, v_2 \rangle - \langle v_1, v_1 \rangle \\
&\Leftrightarrow \langle v_2 - v_1, x \rangle \leq \frac{1}{2}(\|v_2\|^2 - \|v_1\|^2) \\
&\Leftrightarrow c^T x \leq d, \text{ where } c = v_2 - v_1, d = \frac{1}{2}(\|v_2\|^2 - \|v_1\|^2)
\end{aligned}
$$

4. Let A be an $n \times m$ real matrix, and B a $k \times m$ real matrix. Suppose that for every $x \in \mathbb{R}^m$, $Ax = 0$ only if $Bx = 0$. That is,

$$Ax = 0 \Rightarrow Bx = 0.$$

Show that there exists a $k \times n$ matrix C such that $CA = B$.

Here are two ways to do this problem. First, we do it in terms of linear operators: By assumption, $null(A) \subseteq null(B)$, or $dim(null(A)) \leq dim(null(B))$. Let $\{v_1, \ldots, v_{l_1}\}$ be a basis for $null(A)$. Extend this to form a basis for $null(B)$: $\{v_1, \ldots, v_{l_1}, w_1, \ldots, w_{l_2}$. Finally, extend this to form a basis for $\mathbb{R}^m$: $\{v_1, \ldots, v_{l_1}, w_1, \ldots, w_{l_2}, u_1, \ldots, u_{l_3}\}$. Note that we must have $l_1 + l_2 + l_3 = m$. Now, define:

$$x_i = Bu_i, \quad y_i = Au_i \; i = 1, \ldots, l_3.$$

Note that $x_i \in \mathbb{R}^k$ and $y_i \in \mathbb{R}^n$. Also define $z_i = Aw_i$, for $i = 1, \ldots, l_2$. Note that by assumption, the $\{x_i\}$ are independent. Also, the $\{y_i, z_j\}$ are also independent (otherwise some linear combination of the u_i would be in the null space of B, which is not possible, and similarly for a linear combination of w_j, u_i in the null space of A). Because of the independence of vectors $\{y_i, z_j\}$ in $\mathbb{R}^n$, we can extend them to form a basis for $\mathbb{R}^n$, by adding some vectors $\{r_d\}$. So we have a basis for $\mathbb{R}^n$ made up of vectors $\{y_i\}$ (these are the image under A of the independent vectors u_i), the vectors $\{z_j\}$ (these are the image under A of the independent vectors w_j) and then additional vectors $\{r_d\}$ to complete the basis.

Now, use this basis of $\mathbb{R}^n$ to define a linear operator $C : \mathbb{R}^n \to \mathbb{R}^k$ as follows: $Cy_i = x_i$, for $i = 1, \ldots, l_3$, and $Cz_i = 0$, for $i = 1, \ldots, l_2$. This is valid, because the vectors y_i, z_j are independent. If, say, y_1 could be expressed as a linear combination of the vectors z_j, then we would have problems (can you see why?). Finally, define $Cr_d = 0$.

The claim is that the linear operator defined, satisfies $CA = B$. We can check this directly. Take any vector $v \in \mathbb{R}^m$. Then, we can express v in terms of the basis we had above for $\mathbb{R}^m$, i.e., in terms of the vectors $\{v_1, \ldots, v_{l_1}, w_1, \ldots, w_{l_2}, u_1, \ldots, u_{l_3}\}$, and we have:

$$
\begin{aligned}
CA(v) &= CA(\sum_i \alpha_i v_i + \sum_j \beta_j w_j + \sum_r \gamma_r u_r) \\
(\text{using } Av_i = 0, Aw_j = z_j, Au_r = y_r) &= \sum_j \beta_j C(z_j) + \sum_r \gamma_r C(y_r) \\
(\text{using } Cy_r = x_r, C(z_j) = 0) &= \sum_r \gamma_r x_r \\
(\text{using } Bu_r = x_r) &= 0 + 0 + \sum_r \gamma_r B(u_r) \\
(\text{using } Bv_i = Bw_j = 0) &= \sum_i \alpha_i B(v_i) + \sum_j \beta_j B(w_j) + \sum_r \gamma_r B(u_r) \\
&= B(\sum_i \alpha_i v_i + \sum_j \beta_j w_j + \sum_r \gamma_r u_r) \\
&= B(v).
\end{aligned}
$$

This is what we wanted to show.

Here's another way to prove it, more directly through the matrices. If $Ax = 0$, it means that x is perpendicular to every row of A, so that $\langle x\,,\, a_i\rangle = 0$, for each row a_i of A. Therefore (check this for yourselves) $x \in (\text{span}\{a_1, \ldots, a_n\})^{\perp}$. The hypothesis

$$Ax = 0 \Rightarrow Bx = 0,$$

therefore, is equivalent (check this) to the inclusion:

$$(\text{span}\{a_1, \ldots, a_n\})^{\perp} \subseteq (\text{span}\{b_1, \ldots, b_k\})^{\perp}.$$

But this in turn, is equivalent to (check this!)

$$(\text{span}\{a_1, \ldots, a_n\}) \supseteq (\text{span}\{b_1, \ldots, b_k\}).$$

But if the span of the rows of B is contained in the span of the rows of A, then certainly each row of B can be expressed as a linear combination of the rows of A. This is exactly the statement:

$$\exists C \text{ such that } CA = B.$$