The University of Texas at Austin
Department of Electrical and Computer Engineering

EE380K: Linear Systems Theory—Fall 2008

Problem Set Two

C. Caramanis
Due: Wednesday, September 17, 2008.

This problem set continues our development of the tools from linear algebra which we will be using throughout the rest of the semester.

1. Various properties of orthogonal subspaces: Let V be a finite dimensional vector space with an inner produce, and let $U \subseteq V$ be a subspace. Recall that the space $U^{\perp}$ is defined as:

$$U^{\perp} = \{v \in V \,:\, \langle v \,,\, u \rangle = 0, \ \forall u \in U\}.$$

   (a) Show that if U is a subspace, then so is $U^{\perp}$.

   (b) Show that $(U^{\perp})^{\perp} = U$.

   (c) Show that if $U, W \subseteq V$ are subspaces of V, then

$$U \subseteq W \ \Leftrightarrow U^{\perp} \supseteq W^{\perp}.$$

   (d) Suppose now that $X \subseteq V$ is just a subset, i.e., not necessarily a subspace of V. Show that the definition $X^{\perp}$ still makes sense, and that $X^{\perp}$ is a subspace. Next show that $(X^{\perp})^{\perp} \supseteq X$, and it is defined as the smallest subspace that contains the set X.

   (e) Show that when U is a subspace of V, then V is the direct product of U and $U^{\perp}$ (denoted $V = U \oplus U^{\perp}$). That is, show that any $v \in V$ can be written *uniquely* as

$$v = u + u^{\perp},$$

   where $u \in U$, and $u^{\perp} \in U^{\perp}$.

2. Unitary Matrices. Recall from class today that a square matrix U is unitary if its columns form an orthonormal set. This is equivalent (convince yourself) to the condition that its rows form an orthonormal set. In class we said that one property of a unitary matrix, is that it preserves inner products, and in particular, it preserves the 2-norm of a vector: $||Uv||_2 = ||v||_2$.

   Now, recall that matrices A and B are called *similar* if there is a similarity transformation taking A to B, i.e., there exists a matrix S with $A = S^{-1}BS$.

   If S is unitary, then we say that A and B are *unitarily equivalent*. Show that unitary equivalence preserves the Frobenius norm, i.e., $||A||_F = ||B||_F$. (Hint: Use the fact that that $||A||_F^2 = \sum_{i,j} |a_{ij}|^2 = \text{trace}(A'A)$).

3. In class we stated that for a matrix $A \in \mathbb{C}^{m \times n}$,

$$\begin{aligned} ||A||_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \\ ||A||_\infty &= \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|. \end{aligned}$$

   We showed the latter equality. Do the same for the first.

4. Consider two matrices, $A, B \in \mathbb{C}^{n \times n}$.
   (a) Suppose that they are simultaneously diagonalizable, i.e., there exists some S such that $A = S^{-1}\Lambda_A S$, and $B = S^{-1}\Lambda_B S$, where Λ_A and Λ_B are diagonal matrices. Show that A and B commute.
   (b) Next, suppose that A and B are not quite diagonalizable, but they are simultaneously similar to upper triangular matrices, i.e., there exists an S such that $A = S^{-1}\Delta_A S$, and $B = S^{-1}\Delta_B S$, where Δ_A and Δ_B are upper triangular matrices. Show that every eigenvalue of $(AB - BA)$ must be zero.

5. In class we discussed the least-squares problem, of finding a value x to minimize $||Ax - y||_2$. Let us now consider a twist on this problem. Suppose that the value of A and y is not precisely known. Instead, let us supposed that while we see A and y, the true values are

$$A_t = A + \Delta A, \quad y_t = y + \Delta y.$$

   Further, suppose we know that $||\Delta A \ \Delta y||_F \leq 1$, where F denotes the Frobenius norm. Fix a solution, x. How much worse does x do in the worst case, and in the case where there is no noise, i.e., $\Delta A = 0$ and $\Delta y = 0$? That is, compute:

$$\max_{||\Delta A \ \Delta y||_F \leq 1} ||(A + \Delta A)x - (y + \Delta y)||_2.$$

6. Exercise 3.1 from the course notes.

7. Exercise 4.6 from the course notes.

8. Exercise 4.7 from the course notes.