The University of Texas at Austin
Department of Electrical and Computer Engineering

EE380K: Linear Systems Theory—Fall 2008

Problem Set Four

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This problem set focuses on the solution to LTI systems, Jordan form, and other concepts from linear algebra. As usual, any problems marked by a ($*$) are optional (bonus) problems.

1. One more question on SVD: We showed in a past problem set, that the notions of eigenvector and eigenvalue, and also determinant, are invariant to the representation of a linear transformation. That is, if S is invertible, then A and $\hat{A} \triangleq S^{-1}AS$ have the same eigenvalues, and determinant (and also trace). (Of course, the eigenvectors are not the same, but if v is an eigenvector of A, then $S^{-1}v$ is an eigenvector of $S^{-1}AS$.)

   Is σ_1 invariant under similarity transformation? In particular, is our definition of the induced 2-norm invariant under similarity transformation? Why is this ok?

2. Prove that $e^{(N+M)t} = e^{Nt}e^{Mt}$ if N and M commute (i.e., $NM = MN$). You may find the following steps a helpful approach.

   (a) First, show that e^{Nt} and M commute.

   (b) Let $Q(t) = e^{Nt}e^{Mt}$. Take derivatives directly to show that
   $$\frac{d}{dt}Q(t) = (N+M)e^{Nt}e^{Mt}.$$

   (c) If $Q(0) = I$, and
   $$\frac{d}{dt}Q(t) = (N+M)Q(t),$$
   show that we also have:
   $$Q(t) = e^{(N+M)t}.$$

3. Exercise 11.1 from the course notes.

4. Exercise 11.2 from the course notes.

5. Exercise 11.4 from the course notes.

6. Consider a cart of mass M, that moves without friction on a horizontal surface. Its motion is due to an external force, and it can be described by Newton's second law: Force = mass $\times$ acceleration, or
   $$M\frac{d^2y}{dt^2} = u,$$
   where $y(t)$ is the position at time t, and $u(t)$ is the force applied at time t.

   Suppose that we are interested in moving the cart to position $y = 0$. Let's explore some different possible control laws:

(a) First, write a state-space representation of this system, i.e.,

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du.\end{aligned}$$

(b) One simple idea is to apply a force, u, proportional to the displacement from the origin, so that if the cart is at position $y(t) > 0$, we apply force $u(t) = -K_p y(t)$.

Simulate the performance (i.e., examine the trajectories $y(t)$) of the system under this control law. Start with initial conditions $y(0) = 1$, and $\dot{y}(0) = 0$. Try some different values of K_p, such as 0.1, 1, and 10.

Explain what you see, mathematically. That is, if you believe the control law is good, explain why we should expect this to be so. If the control law is no good, explain again what is going on.

(c) Now let's try something more sophisticated: Let u be proportional to the velocity, rather than the position:

$$u(t) = -K_v \dot{y}(t).$$

Again, simulate in Matlab the trajectories of $y(t)$, for $(y(0), \dot{y}(0)) = (0, 1)$, and $K_v = 0.1$, 1, and 10. Can you explain what you see? Now take $K_v = 100$, but use the initial condition $(y(0), \dot{y}(0)) = (1, 1)$. Now what do you see? Again, explain this mathematically.

(d) Now let's combine the two above ideas. Let

$$u(t) = -K_p y(t) - K_v \dot{y}(t).$$

Simulate for different values of K_p, and K_v, and explain mathematically what you observe.

7. $*$ Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. Recall that by the spectral theorem, A will have real eigenvalues. Therefore we can order the eigenvalues of A: $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. Show that:

$$\lambda_k(A) = \max_{V:\dim V = k} \min_{0 \neq y \in V} \frac{\langle y, Ay\rangle}{\langle y, y\rangle}.$$

8. $*$ The spectral radius of a matrix A is defined as:

$$\rho(A) \stackrel{\triangle}{=} \max\{|\lambda| \,:\, \lambda \text{ an e-value of } A.\}.$$

Note that the spectral radius is invariant under similarity transformations, and thus we can speak of the spectral radius of a linear operator.

(a) Show that $\rho(A) \leq \sigma_1(A)$.

(b) Show that

$$\rho(A) = \inf_{\{S:\,\det S \neq 0\}} \sigma_1(S^{-1}AS).$$