The University of Texas at Austin
Department of Electrical and Computer Engineering

EE380K: Linear Systems Theory—Fall 2008

SOLUTIONS FOR PROBLEM SET SIX

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1. Consider the CT LTI state space representation

$$\dot{x} = Ax + Bu,$$

where

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Find a change of basis so that the resulting system is in controllability form; that is,

$$\hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \hat{B}_1 \\ 0 \end{pmatrix}$$

where the pair $(\hat{A}_{11}, \hat{B}_1)$ is controllable.
The controllability matrix is given by:

$$R = [A^3B|A^2B|AB|B] = \begin{pmatrix} 2 & 0 & 2 & 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Clearly, R has rank 2, and we'll use two of the independent columns to form a new basis. Let $t_1 = (1,0,0,0)^\top$ and $t_2 = (1,1,1,0)^\top$. We can easily guess two more vectors orthogonal to these two: $t_3 = (0,0,0,1)^\top$, $t_4 = (0,1,-1,0)^\top$. Now, the change of basis onto these coordinates is given by:

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & -1/2 & 0 \end{pmatrix}$$

Now, changing basis:

$$\hat{A} = T^{-1}AT = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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[1] Solutions written in whole or in part by Johnson Carroll (and Dahleh, et al).

which is precisely the form we wanted. Let's verify that $(\hat{A}_{11}, \hat{B}_1)$ is controllable:

$$\hat{R} = [\hat{A}_{11}\hat{B}|\hat{B}] = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$\hat{R}$ has rank 2, so the pair is controllable. Goodie.

2. **Exercise 22.1** Suppose x_1 is a reachable state, i.e., $x_1 = R_k U_{1_k}$ for some

$$U_{1_k} = \begin{bmatrix} u_1(0) & u_1(1) & \dots & u_1(k-1) \end{bmatrix}'$$

where $u_1(i) \in [0,1]$ and $R_k = \begin{bmatrix} A^{k-1}B & A^{k-2}B & \dots & B \end{bmatrix}$. Suppose that x_1 is in the half-plane H_1 where $H_1 = \{x \in \mathbb{R}^n | w'x \geq 0\}$ and $w'\lambda_1 = w'A$. We want to show that if x_2 is a reachable state, then $x_2 \in H_1$.

First, note that

$$w'x_1 = w' \sum_{i=0}^{k-1} A^{k-i-1}Bu_1(i).$$

Furthermore, $w'A = w'\lambda_1$, so, $w'A^k = w'\lambda_1^k$, and

$$w'x_1 = \sum_{i=0}^{k-1} \lambda_1^{k-i-1} w'Bu_1(i) = \left[\sum_{i=0}^{k-1} \lambda_1^{k-i-1} u_1(i)\right] w'B.$$

Now, since $x_1 \in H_1$ we have that $w'x_1 \geq 0$; and, since $\lambda_1 \geq 0$ and $u_1(i) \geq 0$ we must have that $w'B \geq 0$.

If x_2 is reachable, then $x_2 = R_l U_{2_l}$, where $u_2(i) \in [0,1]$. So,

$$w'x_2 = w' \sum_{i=0}^{l-1} A^{l-i-1}Bu_2(i) = \left[\sum_{i=0}^{l-1} \lambda_1^{l-i-1} u_2(i)\right] w'B.$$

Since $w'B \geq 0$, $\lambda_1 \geq 0$, and $u_2(i) \geq 0$, we have that $w'x_2 \geq 0$, so $x_2 \in H_1$. Finally, we note that there was no loss in generality in assuming $x_1 \in H_1$, as opposed to $x_1 \in H_2 = \{x \in \mathbb{R}^n | w'x \leq 0\}$, because it can be shown that if $x_1 \in H_2$ then $x_2 \in H_2$, and the proof proceeds as above.

3. **Exercise 22.2** a) Given :

$$\begin{aligned} x(k+1) &= \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} x(k) + \begin{pmatrix} d \\ e \end{pmatrix} u(k) \\ &= Ax(k) + Bu(k), \end{aligned}$$

where a, b, c, d, and e are scalar.
In order for this system not to be reachable, the reachability matrix, $\mathcal{C}$ does not have a full rank, which translates to the fact that the determinant of $\mathcal{C}$ be zero. For this case,

$$\begin{aligned}
\mathcal{C} &= \begin{pmatrix} B & AB \end{pmatrix} \\
&= \begin{pmatrix} d & ad+be \\ e & ce \end{pmatrix} \\
\rightarrow det\ \mathcal{C} &= dce - (ad+be)e = 0 \\
\therefore e(d(c-a)-be) &= 0.
\end{aligned}$$

The last equality is the condition.
The block diagram corresponding to the given system is shown below.

(a) $e = 0$
In this case, x_2 is not affected by the control input at all. Thus, x_2 is not reachable.

(b) $b = 0$ and $d = 0$
In this case since $b = 0$, there is no coupling from x_2 to x_1, and x_1 is independent of the control input. Thus x_1 is not reachable.

(c) $b = 0$ and $c = a$.
In this case, there is not coupling between x_1 and x_2. Also, because of $c = a$, the states only can evolve in the direction of input, i.e., $(d, e)^T$. Hence the states can not reach $\mathbb{R}^2$, but just a line whose direction is B matrix direction.

4. **Exercise 22.3** a) The modal test is the most convenient in this case. The system is reachable if and only if rank $[\lambda I - A|B] = 5\ \forall\lambda$ (need to check for λ equal to the eigenvalues of A). Observe that when $\lambda = 2$, $[\lambda I - A|B]$ is

$$\begin{bmatrix} 0 & -1 & & & & b_1 \\ & 0 & & & & b_2 \\ & & 0 & & & b_3 \\ & & & -1 & -1 & b_4 \\ & & & & -1 & b_5 \end{bmatrix},$$

which has rank 5 if and only if b_2 and b_3 are linearly independent. Similarly, $\lambda = 3$, $[\lambda I - A|B]$ is

$$\begin{bmatrix} 1 & 1 & & & & b_1 \\ & 1 & & & & b_2 \\ & & 1 & & & b_3 \\ & & & 0 & 1 & b_4 \\ & & & & 0 & b_5 \end{bmatrix},$$

which has rank 5 if and only if $b_5 \neq 0$.

(b) Suppose that $A \in \mathbb{R}^{n\times n}$ has k Jordan blocks of dimensions (number or rows) r_1, r_2, $\ldots$, r_k. Then we must have that $b_{r_1}, b_{r_1+r_2}, \ldots b_{r_1+r_2+r_3+\cdots+r_k} \neq 0$. Furthermore, if blocks r_i and r_j have the same eigenvalue, $b_{r_1+r_2+r_3+\cdots+r_i}$ and $b_{r_1+r_2+r_3+\cdots+r_j}$ must be linearly independent. These conditions imply that the input can excite the beginning of each Jordan chain, and hence has an impact on each of the states.

(c) If the $b_{i's}$ are scalars, then they are linearly dependent (multiples of each other), so if two of the Jordan blocks have the same eigenvalues the rank of $[\lambda I - A|B]$ is less than n.

Alternatively,

a) The system is reachable if none of the left eigenvectors of matrix A are orthogonal to B. Notice that to control the states corresponding to a Jordan block, it is sufficient to excite only the state corresponding the beginning of the Jordan chain, or the last element in the Jordan block (convince yourself of this considering a DT system for example). Thus it is not necessary that generalized eigenvectors are not orthogonal to the B matrix! Besides, notice that if two or more Jordan blocks have the same eigenvalue than any linear combination of eigenvectors corresponding to those Jordan blocks is a left eigenvector again. In case (a) we can identify left eigenvectors of matrix A:

$$\begin{aligned} w_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}' \\ w_3 &= \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \end{bmatrix}' \\ w_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}' \end{aligned}$$

Any linear combination of w_2 and w_3 is also a left eigenvector. We can see that $w_k'B = b_k'$ - k^{th} row of matrix B. Therefore for reachability of matrix A we need to have at least one non-zero element in 5^{th} row and linear independence of 2^{rd} and 3^{rd} rows of matrix B.

b) Generalizing to an arbitrary matrix in Jordan form we can see that all rows of matrix B corresponding to a Jordan block with unique eigenvalue should have at least one non-zero element, and rows corresponding to Jordan blocks with repeated eigenvalues should be linearly independent.

c) If there are two or more Jordan blocks then we can find a linear combination of the eigenvectors which is orthogonal to the vector b, since two real numbers are obviously linearly dependent.

5. **Exercise 23.3** Given LTI model $\dot{x} = Ax + Bu$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega_2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

a) In order to check asymptotic stability of a LTI system it is suffice to check the eigenvalues of A:

$$\begin{aligned} det(\lambda I - A) &= \begin{vmatrix} \lambda & -1 & 0 & 0 \\ -3\omega^2 & \lambda & 0 & -2\omega \\ 0 & 0 & \lambda & -1 \\ 0 & 2\omega & 0 & \lambda \end{vmatrix} \\ &= \lambda^2(\lambda^2 - \omega^2) = 0. \end{aligned}$$

Thus

$$\lambda = 0, 0, \pm\omega.$$

Since two of the ω's are zero, this system is not asymptotically stable.

b) In order to show that the system is reachable, we would like to check the rank of $\begin{pmatrix} sI - A & \vdots & B \end{pmatrix}$ for all $s \in \mathbb{C}$:

$$\begin{pmatrix} sI - A & \vdots & B \end{pmatrix} = \begin{pmatrix} s & -1 & 0 & 0 & \vdots & 0 & 0 \\ -3\omega^2 & s & 0 & -2\omega & \vdots & 1 & 0 \\ 0 & 0 & s & -1 & \vdots & 0 & 0 \\ 0 & 2\omega & 0 & s & \vdots & 0 & 1 \end{pmatrix},$$

which clearly has rank 4 even when $s = 0$. Therefore this system is reachable.

c) If the radial thruster, $u_2(t)$ fails, then the B matrix is modified to

$$B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T.$$

Then the matrix, $\begin{pmatrix} sI - A & \vdots & B \end{pmatrix}$, is modified correspondingly to

$$\begin{pmatrix} sI - A & \vdots & B_2 \end{pmatrix} = \begin{pmatrix} s & -1 & 0 & 0 & \vdots & 0 \\ -3\omega^2 & s & 0 & -2\omega & \vdots & 0 \\ 0 & 0 & s & -1 & \vdots & 0 \\ 0 & 2\omega & 0 & s & \vdots & 1 \end{pmatrix},$$

which still maintains full rank of 4 for all s. Therefore the system is still reachable.

d)Now if the tangential thruster fails, then a new B matrix is

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T.$$

Thus the matrix $\begin{pmatrix} sI - A & \vdots & B \end{pmatrix}$, is modified correspondingly to

$$\begin{pmatrix} sI - A & \vdots & B_1 \end{pmatrix} = \begin{pmatrix} s & -1 & 0 & 0 & \vdots & 0 \\ -3\omega^2 & s & 0 & -2\omega & \vdots & 1 \\ 0 & 0 & s & -1 & \vdots & 0 \\ 0 & 2\omega & 0 & s & \vdots & 0 \end{pmatrix},$$

which will lose rank to 3 when $s = 0$.

6. **Exercise 23.4** Given :

$$\dot{x}(t) = Ax + (b+\delta)u,$$

where $\delta \in \mathbb{R}^n$, and (A, b) is reachable.
a) Using the Theorem 22.2, in order to make the system unreachable, we have $w^T B = 0$ for some left eigenvectors w^T of A. So, let λ_i be an eigenvalue of A and w_i be the corresponding left eigenvectors. Then, using the theorem, we want to find δ which makes this eigenmode unreachable $\leftrightarrow$ $w_i^T(b+\delta) = 0$. So, now we have

$$w_i^T \delta = -w_i^T b.$$

Then with this constraint, we would like to minimize $\|\delta\|_2$. Thus this can be cast into an optimization problem as follows:

$$\begin{array}{rl} \text{Find} & min\|\delta\|_2 \\ s.t. & w_i^T \delta = -w_i^T b. \end{array}$$

This is exactly in the form of the least square problem. Since both δ and b are real, even when $w_i \in \mathbb{C}^n$, let $\tilde{w}_i = \left[\begin{array}{cc} w_i^R & w_i^I \end{array}\right]$, where w_i^R and w_i^I are real and imaginary parts of w_i respectively. Then the formulation still remains as a least square problem as follows:

$$\begin{array}{rl} \text{Find} & \|\delta\|_2 \\ s.t. & \tilde{w}_i^T \delta = \tilde{w}_i^T b. \end{array}$$

Then the solution to this problem is

$$\begin{array}{rcl} \hat{\delta} & = & -\tilde{w}_i(\tilde{w}_i^T\tilde{w}_i)^{-1}\tilde{w}_i^T b \\ \therefore \min\|\delta\|_2 & = & \sqrt{\hat{\delta}^T\hat{\delta}} \end{array}$$

The last expression has to be minimized over all possible left eigenvectors of A. Note that the expression does not depend on the norm of the eigenvectors, thus we can minimize over eigenvectors with unity norm. If all Jordan blocks of matrix A have different eigenvalues, this is a minimization over a finite set. In the other case we can represent eigenvectors corresponding to Jordan blocks with the same eigenvalues as a linear combination of eigenvectors corresponding to particular Jordan blocks, and then minimize over the coefficients in the linear combination. But note that if we only have a single input, then as we showed in a previous exercise, multiple Jordan blocks with the same eigenvalue necessarily imply the system is not reachable. This need not be the case for multi-input systems.

b) NO. The explanation is as follows. With the control suggested, the closed loop dynamics is now

$$\begin{array}{rcl} \dot{x} & = & Ax + (b+\delta)u \\ u & = & f^T x + v \\ \rightarrow \dot{x} & = & (A + (b+\delta)f^T)x + (b+\delta)v. \end{array}$$

Suppose that w_i was the minimizing eigenvector of unity norm in part a). Then it is also an eigenvector of matrix $A + (b + \delta)f^T$ since w_i is orthogonal to $b + \delta$. Therefore feedback does not improve reachability.

7. **Exercise 23.5** a) First convert the second order scalar differential equation to a first order vector differential equation (Cauchy form):

$$\dot{x} = Ax + bu \ , \ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ , \ b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

The solution is expressed by:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}bu(\tau)d\tau$$

Calculate exponent of matrix A by summing up the series and taking into account that $A^n = 0 \ , \ \forall n > 1$.

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Applying initial conditions $x(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and the above expression for exponent we get the solution:

$$\begin{aligned} x(t) &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} u(\tau)d\tau \ , \ \rightarrow \\ x(t) &= \begin{bmatrix} t \\ 1 \end{bmatrix} - \int_0^t \begin{bmatrix} t-\tau \\ 1 \end{bmatrix} u(\tau)d\tau \end{aligned}$$

b) Compute reachability matrix:

$$\begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The reachability matrix has rank 2, therefore the system is reachable. Now, compute reachability Grammian at time T:

$$P = \int_0^T e^{A(T-\tau)}bb'e^{A(T-\tau)'}d\tau = \int_0^T \begin{bmatrix} T-\tau \\ 1 \end{bmatrix}\begin{bmatrix} T-\tau & 1 \end{bmatrix} d\tau = \begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} \\ \frac{T^2}{2} & T \end{bmatrix}$$

c) We would like to solve the following problem:

$$\min \int_0^T u^2 d\tau, \text{ with the constraint that: } -\begin{bmatrix} T \\ 1 \end{bmatrix} = -\int_0^T \begin{bmatrix} T-\tau \\ 1 \end{bmatrix} u(\tau)d\tau$$

Denote

$$a_1 = -(T-t) \ , \ a_2 = -1 \ , \ f = -\begin{bmatrix} T \\ 1 \end{bmatrix}$$

and introduce regularL_2 scalar product:

$$< a, b >= \int_0^T (a \cdot b)dt$$

Then the problem can be reformulated as follows:

$$\min_{<a_k,u>=f_k} < u, u > \ , \ k = 1, 2$$

Functions $a_1 = -(T - t)$ and $a_2 = -1$ are linearly independent (but not orthogonal in this case). We can augment these with $a_m \ , \ m > 2$ to form a complete basis in $L_2 \left[\begin{array}{cc} 0 & T \end{array} \right]$. Besides we can choose $a_m \ , \ m > 2$ orthogonal t each other and to a_1 and a_2. Then the optimal solution can be expressed as a linear combination of basis functions:

$$u = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + ... = u_0 + \tilde{u} \ , \ u_0 = \alpha_1 a_1 + \alpha_2 a_2$$

then the minimization problem becomes:

$$\min_{<a_k,u_0+\tilde{u}>=f_k} < u_0 + \tilde{u}, u_0 + \tilde{u} > \ , \ k = 1, 2$$

Since $< a_k, \tilde{u} >= 0 \ , \ k = 1, 2$ by construction we can rewrite it as

$$\min_{<a_k,u_0>=f_k} < u_0, u_0 > + < \tilde{u}, \tilde{u} > \ , \ k = 1, 2$$

Since there are no constraints on $\tilde{u}$ we can set it equal to zero. Thus optimal control function is a linear combination of a_1 and a_2 (recall projection theorem!). Therefore constraints give us two equations for the two unknown coefficients:

$$\left[\begin{array}{cc} < a_1, a_1 > & < a_1, a_2 > \\ < a_2, a_1 > & < a_2, a_2 > \end{array} \right] \left[\begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right] = \left[\begin{array}{c} f_1 \\ f_2 \end{array} \right]$$

It is easy to see that the matrix composed of scalar products of a_1 and a_2 is exactly the reachability Grammian of the system. Since the system is reachable it is positive definite and therefore invertible. Calculating the optimal control from the equation above we get:

$$u(t) = -f' P^{-1} e^{A(T-t)} b = -\frac{6t}{T^2} + \frac{4}{T}$$

d) For $T = 0.001$

$$u = -\frac{6t}{0.001^2} + \frac{4}{0.001}$$

and its plot is given in Figure 1 below. We can see that the optimal control is linear with time, positive in beginning (deceleration), and switches to negative (acceleration) when the velocity reaches its minimum. Note that this problem falls into a class of optimal control problem of LTI systems with a quadratic cost function and has a closed form solution with linear, but time varying, state feedback. More about it in 6.245.

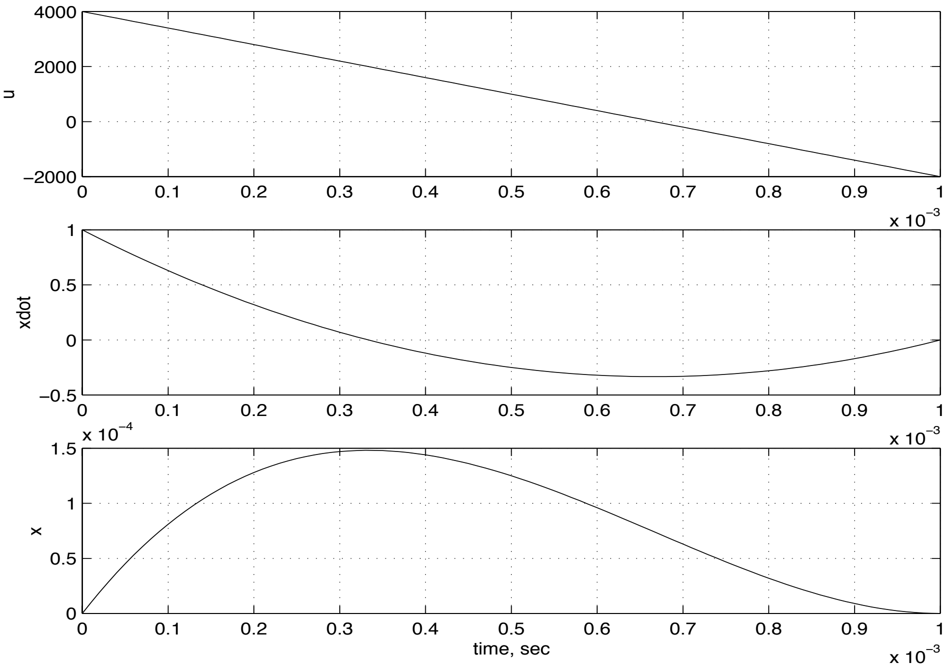

Figure 1: 23.5: Optimal control and optimal trajectory.