The University of Texas at Austin
Department of Electrical and Computer Engineering

EE380K: Linear Systems Theory—Fall 2008

Solutions for Problem Set Seven

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1. **Exercise 24.2** (a) To show the system is observable, we calculate the observability matrix at time k, $\mathcal{O}_k$:

$$\mathcal{O}_k = \begin{bmatrix} C_1 & 0 & 0 & 0 & \dots & 0 \\ * & C_1A_{12} & 0 & 0 & \dots & 0 \\ * & * & C_1A_{12}A_{23} & 0 & \dots & 0 \\ * & * & * & C_1A_{12}A_{23}A_{34} & \dots & 0 \\ & & & & \ddots & \\ * & * & * & * & \dots & C_1A_{12}A_{23}\dots A_{k-1,k} \end{bmatrix}$$

where $*$ denote the entries that result from the matrix multiplication. Now, in general, if P and Q have full column rank, then PQ has full column rank. This follows from Sylvester's inequality (see problem 1.4). So, since C and $A_{i,i-1}$ have full column rank, all the matrices on the diagonal of the matrix $\mathcal{O}_k$ have full column rank; hence, $\mathcal{O}_k$ has full column rank, and therefore $\mathcal{O}_n$, where n is the dimension of the state space (i.e. A is $n \times n$), will also have full column rank (note that $n \geq k$).

(b) If $A_{k-1,k} = 0$ then the observability matrix, $\mathcal{O}_n$ will have zeros in its last m-columns, where m is the number of columns of $A_{k-1,k}$; that is, $\mathcal{O}_n$ drops rank, and the system is unobservable. This is obvious in the case where $k = n$ bacause the observability matrix will have the same structure shown above. Now if $n > k$, the last m-columns of CA^k, CA^{k+1}, $\dots$, CA^{n-1} will still be zero. This can be shown by induction. If $A_{k-1,k} = 0$, then the last m-columns of CA^{k-1} will be zero since the last element of CA^{k-1} is $C_1A_{12}A_{23}\dots A_{k-1,k}$. Now,assume that the last m-columns of CA^i, $k < i < n$ is zero. Then the last m-columns of $CA^{i+1} = (CA^i)A$ is still zero because the last m-columns of A are $\begin{bmatrix} 0 & 0 & \dots & 0 & A_{kk}^T \end{bmatrix}^T$.

2. Because, as discussed in class, observability is the dual of reachability, this proof follows the form of problem 23.4 which you did last week in the homework. Considering $(C+\delta)$ and using right eigenvectors instead of left, etc, produces the appropriate result with few modifications to the analysis of problem 23.4.

3. **Exercise 24.5** a) The given system in general for all $t \geq 0$ with $u(k) = 0$ $\forall k \geq 0$ has the following expression for the output:

[1] Solutions written in whole or in part by Johnson Carroll (and Dahleh, et al).

$$
\begin{aligned}
y(t) &= \Sigma_{k=-1}^{-\infty} CA^{t-k-1}Bu(k) \\
&= CA^{t}\Sigma_{k=-1}^{-\infty} A^{-k-1}Bu(k)
\end{aligned}
$$

since matrix A is stable. Note that because of stability of matrix A all of its eigenvalues are strictly within unit circle, and from Jordan decomposition we can see that

$$
\lim_{k\to\infty} \|A^k\|_2 = 0
$$

therefore $x(-\infty)$ does not influence $x(0)$. Thus the above equation can be used in order to find $x(0)$ as follows:

$$
x(0) = \sum_{k=-1}^{-\infty} A^{-k-1}Bu(k).
$$

b) Since the system is reachable, any $\xi \in \mathbf{R}^n$ can be achieved by some choice of an input of the above form. Also, since the system is reachable, the reachability matrix $\mathcal{R}$ has full row rank. As a consequence $(\mathcal{R}\mathcal{R}^T)^{-1}$ exists. Thus, in order to minimize the input energy, we have to solve the following familiar least square problem:

$$
\begin{aligned}
\text{Find} \qquad & \min \|u\|_2 \\
s.t. \qquad & \xi = \sum_{k=-1}^{-\infty} A^{-k-1}Bu(k).
\end{aligned}
$$

Then the solution can be written in terms of the reachability matrix as follows:

$$
u_{min} = \mathcal{R}^T(\mathcal{R}\mathcal{R}^T)^{-1}\xi,
$$

so that its square can be expressed as

$$
\begin{aligned}
\|u\|_{min}^2 &= u_{min}^T u_{min} \\
&= \xi^T(((\mathcal{R}\mathcal{R}^T)^{-1})^T\mathcal{R}\mathcal{R}^T(\mathcal{R}\mathcal{R}^T)^{-1}\xi \\
&= \xi^T(\mathcal{R}\mathcal{R}^T)^{-1}\xi,
\end{aligned}
$$

where the last equality comes from the fact that inverse of a symmetric positive definite matrix is still symmetric positive definite. Also, the Controlability Gramian of DT systems $\mathcal{P}$ is

$$
\mathcal{P} = \sum_{k=0}^{\infty} A^k BB^T(A^T)^k = \mathcal{R}\mathcal{R}^T,
$$

and is symmetric positive definite. Thus the square of the minimum energy, denoted as $\alpha_1(\xi)$, can be expressed as

$$\alpha_1(\xi) = \xi^T \mathcal{P}^{-1} \xi = \|M\xi\|_2^2$$

where M is a Hermitian square root matrix of $\mathcal{P}^{-1}$ which is still symmetric positive definite.

c) Suppose some input u_{min} results in $x(0) = \xi$, then the output for $t \geq 0$ can be expressed as

$$y(t) = Cx(t) = CA^t\xi.$$

Thus the square of the energy of the output for $t \geq 0$ can be written as

$$\begin{aligned} \|y\|_2^2 &= (y^T y) \\ &= \left(\begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} \xi \right)^T \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} \xi \\ &= \xi^T \left(\sum_{t=0}^{\infty} (A^T)^k C^T C A^k \right) \xi \\ &= \xi^T \mathcal{O}^T \mathcal{O} \xi \end{aligned}$$

Since the Observability Grammian of DT systems $\mathcal{Q}$ is

$$\mathcal{Q} = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k = \mathcal{O}^T \mathcal{O},$$

the square of the energy of the output for $t \geq 0$, which we now denote $\alpha_2(\xi)$, can be expressed as a function of ξ as follows:

$$\alpha_2(\xi) \equiv \xi^T \mathcal{Q} \xi.$$

Also, because of the symmetric positive definiteness of $\mathcal{Q}$, $\alpha_2(\xi)$ can be written as

$$\alpha_2(\xi) = \|N\xi\|_2^2,$$

where N is a Hermitian square root matrix of $\mathcal{Q}$.

d) It can be argued as follows:

$$
\begin{aligned}
\alpha &= \max_u \{ \sum_{t=0}^{\infty} y(t)^2 \mid \sum_{t=-\infty}^{-1} u(t)^2 \leq 1 \ , \ u(k) = 0 \forall k \geq 0 \} \\
&= \max_\xi \{ \alpha_2(\xi) \mid \exists \ u \text{ s.t.} \xi = x(0) \text{ and } \sum_{t=-\infty}^{-1} u(t)^2 \leq 1 \ , \ u(k) = 0 \ , \ \forall k \geq 0 \} \\
&= \max_\xi \{ \alpha_2(\xi) \mid \|u_{min}\|_2^2 \leq 1 \} \\
&= \max_\xi \{ \alpha_2(\xi) \mid \alpha_1(\xi) \leq 1 \}.
\end{aligned}
$$

e) Now, using the fact shown in d) and noting the fact that $\mathcal{P}^{-1} = M^T M$ where M is Hermitian square root matrix which is invertible, we can compute α:

$$
\begin{aligned}
\alpha &= \max_\xi \{ \alpha_2(\xi) \mid \alpha_1(\xi) \leq 1 \} \\
&= \max_\xi \{ \|N\xi\|_2^2 \mid \|M\xi\|_2^2 \leq 1 \} \ \ \text{set } \xi = M^{-1} l \\
&= \max_l \{ (M^{-1} l)^T \mathcal{O}^T \mathcal{O} M^{-1} l \mid \|l\|_2^2 \leq 1 \} \\
&= \sigma_{max}(\mathcal{O} M^{-1}) \\
&= \lambda_{max}((M^{-1})^T \mathcal{O}^T \mathcal{O} M^{-1}) \\
&= \lambda_{max}((M^{-1})^T \mathcal{Q} M^{-1}) \\
&= \lambda_{max}(\mathcal{Q} M^{-1} (M^{-1})^T) \\
\therefore \alpha &= \lambda_{max}(\mathcal{Q}\mathcal{P})
\end{aligned}
$$