The University of Texas at Austin
Department of Electrical and Computer Engineering

EE380K: Linear Systems Theory—Fall 2008

Problem Set 9

C. Caramanis **Not Due**.

This problem set focuses on the ideas of Lyapunov functions, observability, controllability, and optimal control.

1. We computed the continuous time Lyapunov equation by looking for a quadratic Lyapunov function. Do the same for discrete time.

2. Recall the Hamiltonian matrix we introduced in class. We stated in class that the eigenvalues of $\mathcal{H}$ are symmetric about the imaginary axis, i.e., if λ is an eigenvalue then so is $-\bar{\lambda}$. Prove this, filling in any steps we omitted in class.

3. In class we used the matrix exponential to obtain the solution to the LTI ODE obtained using the Hamiltonian matrix. Then letting $t_1 \to \infty$, we obtained a solution to the ARE: Algebraic Riccati Equation. In this exercise you will do this more directly by proving the following result:

   Suppose V is an n-dimensional invariant subspace of the Hamiltonian matrix $\mathcal{H}$. Recall that this means that if $x \in V$, then $\mathcal{H}x \in V$. Suppose now, that P_1 and P_2 are $n \times n$ matrices, such that
   $$V = \text{Range}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.$$
   Show that if P_1 is invertible, then $\bar{P} = P_2 P_1^{-1}$ is a solution to the ARE. (Hint: Note that you can rewrite the ARE as: $[-\bar{P} \;\; I_{n\times n}]\mathcal{H}[I_{n\times n} \;\; \bar{P}]^\top = 0$.

4. Use the approach of the previous problem to compute solutions to the ARE when we have:
   $$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = 0.$$

5. With the data from the previous problem, compute the optimal constant gain controller for the infinite horizon case, and then compute the resulting closed-loop eigenvalues of the system.

6. In class we showed that for a single input system, if the pair (A, B) is controllable, then it can be brought into so-called reachable-canonical-form:
   $$\tilde{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & & \ddots & \\ \vdots & & & 1 \\ -a_0 & \cdots & \cdots & -a_{n-1} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

   (a) Show the converse statement.

(b) Now for observability: Show that in a single output system, if the pair (A, C) is observable, then there exists a transformation that puts the system into so-called observer-canonical-form:

$$\tilde{A} = \begin{bmatrix} -a_0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ -a_{n-1} & 0 & \cdots & 0 \end{bmatrix}, \qquad \tilde{C} = [\ 1 \quad 0 \quad \cdots \quad 0\ ].$$

7. Consider the following CT LTI system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & -1 \\ 4 & -1 & 2 \end{bmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \\ y &= [\ 1 \quad 0 \quad 0\ ] \end{aligned}$$

Note that we have:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & -1 \\ 4 & -1 & 2 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 9 & -2 & 6 \\ 5 & 5 & 2 \\ 9 & -4 & 13 \end{bmatrix}.$$

(a) Is this system reachable?

(b) Now consider the system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 4 & -5 \\ -3 & 2 \end{bmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= [\ 1 \quad 0\ ] \end{aligned}$$

Compute: (i) the Reachable Canonical Form (RCF), (ii) the Observable Canonical Form (OCF), and (iii) the Jordan Canonical Form (JCF) of the system matrix (i.e., of the A-matrix).

c) Now consider a general CT LTI system of the form:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + Bu \\ y &= C\boldsymbol{x}. \end{aligned}$$

Let $\boldsymbol{z} = T\boldsymbol{x}$ represent a coordinate transformation (so that T is invertible). Then $\boldsymbol{z}$ satisfies a linear system:

$$\begin{aligned} \dot{\boldsymbol{z}} &= \tilde{A}\boldsymbol{z} + \tilde{B}u \\ y &= \tilde{C}\boldsymbol{z}. \end{aligned}$$

Compute $\tilde{A}, \tilde{B}, \tilde{C}$ in terms of A, B, C, and T.

(d) Let V be the states reachable at time T from the origin. Show that V is a vector space.

8. Consider the CT LTI system from the previous problem:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & -1 \\ 4 & -1 & 2 \end{bmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \\ y &= [\ 1 \quad 0 \quad 0\ ] \end{aligned}$$

(a) Compute the Observer Canonical Form of the matrix A.

(b) Let T_1 be the invertible transformation that puts the system into OCF, and T_2 the invertible transformation that puts the system into RCF. *In these new coordinates*, construct a linear (state) feedback law that stabilizes the system and a linear observer that has asymptotically no error, and then give a linear controller, and a linear observer *in the original coordinates* in terms of your previous answer, and the transformation matrices T_1 and T_2.

(c) Now consider the larger CT LTI system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 3 & 2 & -1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -4 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Argue that this system is NOT observable and NOT reachable. [Note that you can do this without computing A, A^2, A^3, A^4.]

(d) *Even though the system is not observable, and not reachable*, construct an observer, and a feedback law that uses the observed state, so that the overall system is stable. Your answer should be in the original coordinate system, but you can express your answer using the matrices T_1 and T_2 from part (b) above.